Astrophysics Formulas

Compact Listings

by astrophysicsformulas.com

Astrophysics Formulas

Note: Items underlined or boxed in red are active hyperlinks which can be clicked on to open a browser window.

This is a list of commonly used formulas in astrophysics, which is by no means exhaustive. The easiest way to find what you are looking for in this pdf file is to use the search facility in your pdf viewer (for example, on a mac it can be accessed by pressing the "apple" and "F" keys respectively). Please visit http://astrophysicsformulas.com for more details and explanation for a particular item, including complete definitions of symbols used. Note that printing is disabled on this document. Please visit http://astrophysicsformulas.com for information about the availability of a printable version. Disclosure: some of the hyperlinks in this document are affiliate links.

A Note About Units

You will notice that a mix of different systems of units are used, and there is often jumping between different systems. This may seem inconsistent and erratic but it is how practising astrophysicists operate. This is usually determined either by convenience or by deep-seated conventions.

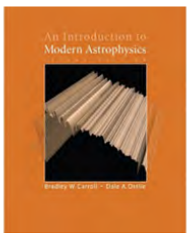
In some cases where a general formula or equation is given, particular units may be shown in the formula or equation. The choice of units is not meant to signify any importance of those units, they are given only for illustration in order to facilitate understanding of a formula or equation. See "Fundamental Constants & Solar System Data" section for sources of data used in this document.

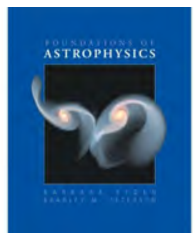
Contents

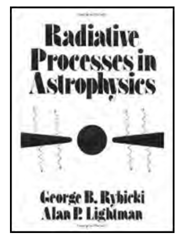
Books

An Introduction to Modern Astrophysics by Bradley W. Carroll and Dale A. Oslie. This book covers an extremely wide range of topics as might be guessed from its colossal length of 1400 pages. The depth is suitable for a core text for undergraduate level and is also suitable as a broad reference for graduate level. See details and reviews at Amazon →.

Foundations of Astrophysics by Barbara Ryden and Bradley M. Peterson. This book is less than half of the length of Carroll and Oslie, and also covers a very broad range of topics. Both books have their strengths and weaknesses. It is recommended that either or both books are supplemented by more practice with solving problems, and by books on specialized topics (such as radiative transfer, stellar structure, cosmology, etc.). See details and reviews at Amazon →.

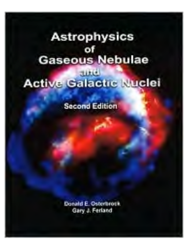


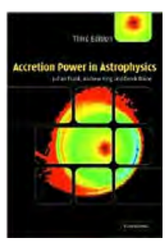


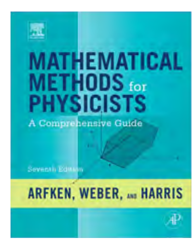


Radiative Processes in Astrophysics by George B. Rybicki and Alan P. Lightman. This is a classic text in radiative transfer. It is suitable for undergraduate level as well as graduate level. The text is concise but often too terse for students who are exposed to the material for the first time. However, supplemented with a more general text and more problem-solving practice, it is an essential component of a personal astrophysics library, at both undergraduate and graduate levels. See details and reviews at Amazon →.

Astrophysics of Gaseous Nebulae by Donald E. Osterbrock and Gary Ferland. A classic text that focuses on physical processes in ionized plasmas and applications in astrophysics geared towards observations of active galactic nuclei. Suitable for both undergraduate and graduate levels. See details and reviews at Amazon →.







Accretion Power in Astrophysics by Juhan Frank, Andrew King, and Derek Raine. A very thorough text the accretion process in X-ray binaries and active galactic nuclei. Bondi accretion is also covered. Suitable for graduate level. See details and reviews at Amazon →.

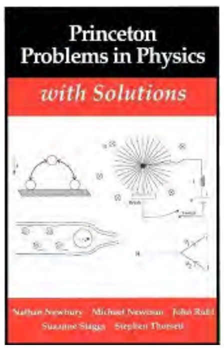
Mathematical Methods for Physicists by George B. Arfken, Hans J. Weber, and Frank E. Harris. This is a classic and widely-used text on mathematical methods in physics and is very thorough with plenty of problems. See details and reviews at Amazon →.

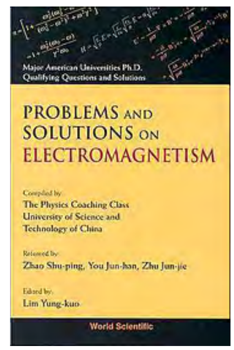
Problems with Solutions

The following books all have problems with worked solutions for essential practice in mastering techniques and enhancing understanding. They have material that is at both the undergraduate and graduate levels which is also excellent for preparing for university entrance exams.

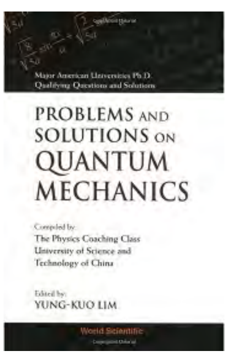
Click on the cover images to go to Amazon detail page and read reviews

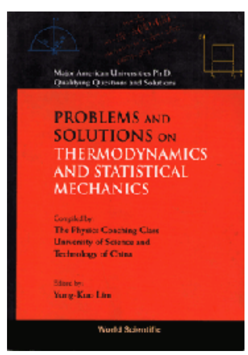
Princeton Problems in Physics with Solutions by Nathan Newbury, John Ruhl, Suzanne Staggs and Stephen Thorsett.
Books by Y. K. Lim (Major American Universities Ph.D. Qualifying Questions and Solutions):
Problems and Solutions on Electromagnetism
Problems and Solutions on Thermodynamics and Statistical Mechanics

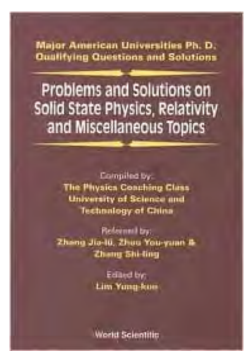




Problems and Solutions on Thermodynamics and Statistical Mechanics
Problems and Solutions on Quantum Mechanics
Problems and Solutions on Solid State Physics, Relativity and Miscellaneous Topics







Astrophysics Formulas

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1 Elementary Relations

Parallax method for distance measurement.

$$\tan\theta \sim \frac{x}{d} \sim \theta \sim \sin\theta$$

The parallax angle, θ, is in radians, and is equal to *half* of the angular shift between two observational positions; x is *half* of the the baseline distance between those observational positions, and d is the distance from the observer to the star (in the same units as x and is the same value for both observational positions).

Parsecs and parallax distance.

$$d = \frac{1}{\mathrm{parallax(in\ arcseconds)}}\ \ \mathrm{pc}$$

Note: 1 parsec (pc) $=3.26163626$ light years, or 3.086×10^{18} cm.

Solid angle.
The solid angle, Ω, subtended by a complete sphere at the center is 4π. The solid angle subtended by an arbitrary area at a point is 4π times the fraction that such an area is of a the complete area of a sphere centered on that point.

$$\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\phi, \quad \Omega = \int_S \sin\theta\mathrm{d}\theta\mathrm{d}\phi$$

Distances to objects with small redshift.

If $z \ll 1$,

$$d \sim \frac{cz}{H_0} = \frac{4163.78z}{h_{72}}\ \mathrm{Mpc} \quad (H_0 \equiv 72h_{72}).$$

See cosmology section for z not $\ll 1$. H_0 is the Hubble constant in units of $\mathrm{km\ s^{-1}\ Mpc^{-1}}$.

Angular distance between two points on a sphere

Suppose that the two points have a RA (right ascension) of ϕ_1 and ϕ_2, and a DEC (declination) of θ_1 and θ_2. Alternatively, the angles ϕ and θ could correspond to geographical longitude and latitude respectively. The angle at the center of the sphere separating the two points is:

$$\Psi = \arccos\left(\sin\theta_1 \sin\theta_2 + \cos\theta_1 \cos\theta_2 \cos\left(\phi_1 - \phi_2\right)\right).$$

The arc length on the spherical surface is equal to the radius of the sphere times Ψ (in radians). When the two points are extremely close together (or otherwise when $|\cos\Psi|$ is close to unity), different formulas must be used to ensure sufficient precision (the *haversine* formula, or the *Vincenty* formula).

Inverse square law for radiation.

$$F_r(\mathrm{erg\ cm^{-2}\ s^{-1}}) = \frac{L(\mathrm{erg\ s^{-1}})}{4\pi[r(\mathrm{cm})]^2}, \quad N_r(\mathrm{photons\ cm^{-2}\ s^{-1}}) = \frac{N_0(\mathrm{photons\ s^{-1}})}{4\pi[r(\mathrm{cm})]^2}$$

$$\frac{F_1}{F_2} = \frac{N_1}{N_2}\left(\frac{r_2}{r_1}\right)^2$$

Conditions: the emission is isotropic and due to a point source; there is no absorption or other obstacle in between emission and reception sites.

Apparent and absolute magnitude.

Absolute magnitude, for a specified waveband interval, must by calibrated (several conventions are in use: see entry at astrophysicsformulas.com), and corresponds to the absolute flux that a star would have at 10 pc.

$$\text{Absolute magnitude} \ \equiv \ M \ = \ -2.5\log F(r = 10\ \mathrm{pc}) + C$$

where C is a constant that depends on the convention adopted to set the scale. Apparent magnitude, m, is the corresponding flux at the actual observer-star separation:

$$m - M \ = \ 5\log_{10}\left(\frac{r}{10\ \mathrm{pc}}\right) \ = \ 5(\log_{10} r - 1).$$

Luminosity ratios: A change in 5 magnitudes corresponds to a change in luminosity by 100 (by definition), so a change of $\Delta m \equiv m_2 - m_1$ magnitudes corresponds to a luminosity ratio of:

$$\begin{aligned} \frac{L_1}{L_2} &= 10^{\frac{2}{5}(m_2 - m_1)} \quad \text{or} \\ m_2 - m_1 &= 2.5(\log L_1 - \log L_2) \\ &= 2.5\log\left(L_1/L_2\right) \end{aligned}$$

Wavelength, frequency, energy, momentum relations for photons

$$\nu\lambda = c$$
$$\omega = 2\pi\nu$$
$$k = |\mathbf{k}| \equiv \frac{2\pi}{\lambda}$$
$$E = h\nu = \hbar\omega$$
$$E = \frac{hc}{\lambda}$$
$$E = pc$$
$$\mathbf{p} = \hbar\mathbf{k}$$

Wavelength, frequency, energy, momentum relations for matter
All of the following are nonrelativistic.

$$E = \frac{p^2}{2m} \quad (= \frac{1}{2}mv^2 = \frac{(mv)^2}{2m})$$
$$\text{de Broglie wavelength} = \frac{h}{p}$$
$$= \frac{h}{\sqrt{2mE}} \text{ (using } E = p^2/2m)$$
$$k = |\mathbf{k}| \equiv \frac{2\pi}{\lambda}$$
$$\mathbf{p} = \hbar\mathbf{k}$$
$$\nu\lambda = v = p/m$$
$$\omega = 2\pi\nu$$
$$\frac{\omega}{k} = v = p/m$$

List of symbols used above:
c: Speed of light.
E: Energy.
h: Planck's constant.
$\hbar$: $h/2\pi$.
$\boldsymbol{k}$: Wave vector.
k: $= |\boldsymbol{k}|$.
λ: wavelength.
m: Mass.
ν: Frequency.
ω: Angular frequency.
$\boldsymbol{p}$: Momentum.
p: $= |\boldsymbol{p}|$.
v: Magnitude of velocity.

Column density

$$N_i = n_i d, \qquad N_{\rm H} = n_{\rm H} d \quad \text{atoms or ions per unit area}$$

where d is the length of matter along the pertinent direction (e.g., line-of-sight) and $n_{\rm i}$ is the volume number density of the atom or ion in question. The second formula is just the specific case of hydrogen.

Mean free path

$$l = \frac{1}{n\sigma}$$

where n is the number density per unit volume of the objects in question, and σ is their cross-sectional area (effective cross-sectional area if necessary).

Optical Depth

$$\begin{aligned}\tau_\nu &= \alpha_\nu\, d \quad (\alpha_\nu = \text{absorption coefficient per unit length})\\ &= n\,\sigma_\nu\, d \quad (n = \text{number density};\ \sigma_\nu = \text{absorption cross-section})\\ &= \rho\,\kappa_\nu\, d \quad (\rho = \text{density};\ \kappa_\nu = \text{mass absorption coeffcient})\\ &= N_{\rm H}\sigma_{\rm E} \quad (N_{\rm H} = \text{column density};\ \sigma_{\rm E} = \text{absorption cross-section})\end{aligned}$$

where d is a length in consistent units; ν is frequency, E is energy.

Thomson electron scattering cross-section
Total cross-section for scattering of photons by free electrons, averaged over all scattering angles:

$$\sigma_T = \frac{8\pi}{3} r_e^2 = \frac{8\pi}{3}\frac{e^4}{m_e^2 c^2} = 6.55 \times 10^{-25}\ \mathrm{cm}^2$$

where $r_e \equiv e^2/(4\pi\epsilon_0 m_e c^2)$ is the classical electron radius. The differential cross-section (for unpolarized radiation) is:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{3\sigma_T}{16\pi}(1+\cos^2\theta)$$

where θ is the angle between the incident and scattered radiation. Thomson scattering is independent of energy but is only valid in the classical regime, for incident photon energies that are $\ll m_e c^2$.

Rayleigh Scattering
Rayleigh scattering refers to photons scattering on electrons that are bound (e.g., in an atom). If the characteristic resonance frequency of the bound state is ν_0, the standard solution for a classical damped oscillator gives:

$$\sigma_{\rm R}(\nu) = \sigma_T \frac{\nu^4}{(\nu^2-\nu_0^2)^2 + (\gamma^2\nu^2/4\pi^2)} \quad \mathrm{cm}^2$$

where $\sigma_T = 6.65 \times 10^{-25}\ \mathrm{cm}^2$, and γ is the classical radiation damping constant. Clearly, for $\nu \ll \nu_0$, $\sigma_{\mathrm{R}} \propto (\nu/\nu_0)^4$. The corresponding quantum mechanical result also has the same frequency dependence (e.g., see H. A. Kramers 1924, Nature, 113, 673).

Mean number of scatterings

$$\begin{aligned} N_{\mathrm{scatterings}} &= \tau_s \quad \text{if } \tau \ll 1 \\ &= \tau_s^2 \quad \text{if } \tau > 1 \end{aligned}$$

in a pure scattering medium, where τ_s is the scattering optical depth. The number refers to the mean number of scatterings of, say, photons, between origin and escape. These are approximations only. The real values depend on geometry and other factors.

Effective optical depth due to absorption and scattering

$$\tau_{\mathrm{eff}} = \sqrt{\tau_{\mathrm{abs}}(\tau_{\mathrm{abs}} + \tau_{\mathrm{s}})}$$

where τ_{abs} and τ_{s} is the absorption and scattering optical depth respectively. This is only an approximation; the correct procedure should account for geometry and other factors (in general, there is no exact analytic solution).

FWHM of a Gaussian line profile
For a Gaussian line profile, $A \exp\left[-(E - E_0)^2/(2\sigma^2)\right]$,

$$\mathrm{FWHM} = [2\sqrt{(2\ln 2}]\sigma = 2.35482\sigma$$

FWHM of a Lorentzian line profile
For a Lorentzian line profile expressed as

$$\begin{aligned} L(\nu) &= A\frac{\gamma}{(\nu - \nu_0)^2 + \gamma^2} \\ \mathrm{FWHM} &= 2\gamma\ . \end{aligned}$$

Thermal Doppler line broadening
The line profile is a Gaussian, $\exp\left[-(\nu - \nu_0)^2/W_D^2\right]/[\sqrt{\pi}W_D]$, where the Doppler width, W_D is *defined* to be $\sqrt{2}\sigma$, and

$$W_D \equiv \nu_0 \left(\frac{2kT}{mc^2}\right)^{\frac{1}{2}}, \quad \text{so} \quad \mathrm{FWHM} = \sqrt{2}W_D = 4\sqrt{\ln 2}\sigma = 3.330\sigma$$

Virial theorem
For a "relaxed" gravitationally bound system of bodies in dynamical equilibrium, the total (time-averaged) kinetic energy (K.E., or T) of the constituents is equal to half of the negative of the total (time-averaged) potential energy (P.E.). The total energy, E, is therefore equal to half of the P.E (U).

$$< \mathrm{T} >= -\frac{1}{2} < \mathrm{U} >, \quad \mathrm{E} = \frac{1}{2} < \mathrm{U} >$$

Mean molecular weight relative to Hydrogen

The mean molecular weight of a gas is the effective mass per particle in units of the mass of an H atom. For species labeled by j, and for n_j atoms with mass m_j,

$$\begin{aligned}\mu &\equiv \frac{\sum_j n_j m_j}{\sum_j n_j} \\ &= f_{\mathrm{H}} + \frac{1}{4} f_{\mathrm{He}} + \frac{1}{A} f_{\mathrm{metals}} \quad \text{(neutral gas)} \\ &= 2 f_{\mathrm{H}} + \frac{3}{4} f_{\mathrm{He}} + f_{\mathrm{metals}} \frac{< 1 + Z >}{< A >} \quad \text{(completely ionized gas)}\end{aligned}$$

The f values are mass fractions of the pertinent species, A is the number of nucleons, and Z is atomic number. Reasonable choice for cosmic abundance: $f_{\mathrm{H}} = 0.7$, $f_{\mathrm{He}} = 0.28$, $f_{\mathrm{metals}} = 0.02$ (but depends on which cosmic abundances exactly). Then $\mu = 1.39$ and 0.62 for neutral and completely ionized gas respectively.

Jean's mass and Jean's length

For interstellar gas with a mean molecular weight of μ, temperature, T, and mean density ρ,

$$\begin{aligned}M_{\mathrm{Jeans}} &\sim \left(\frac{5kT}{G\mu m_{\mathrm{H}}}\right)^{\frac{3}{2}} \left(\frac{3}{4\pi\rho}\right)^{\frac{1}{2}} \\ R_{\mathrm{Jeans}} &\sim \left(\frac{15kT}{4\pi G\mu m_{\mathrm{H}}}\rho\right)^{\frac{1}{2}}\end{aligned}$$

Speed of sound

For a gas with a pressure, P, and a density, ρ, the speed of sound, c_s, is

$$c_s = \sqrt{\frac{\text{bulk modulus}}{\rho}} = \left(\frac{\partial P}{\partial \rho}\right)^{\frac{1}{2}}_{\text{adiabatic}} = \left(\frac{\gamma P}{\rho}\right)^{\frac{1}{2}} = \left(\frac{\gamma k T}{\mu\, m_{\mathrm{H}}}\right)^{\frac{1}{2}}$$

where the last two identities correspond to an ideal gas, with γ equal to the ratio of heat capacity at constant pressure to that at constant volume. For air at 273.15 Kelvin with no humidity, $c_s \sim 331\ \mathrm{m\ s^{-1}}$. Multiply by $(T/273.15)^{\frac{1}{2}}$ for other temperatures (T, in Kelvin).

For a solid, the three-dimensional structure affects the propagation of sound and there are generally two speeds corresponding to longitudinal and shear disturbances. In the long rod case below, there is effectively only one speed.

Continued on next page.

$$
\begin{aligned}
c_{s,L}(\text{longitudinal}) &= \left(\frac{\text{bulk modulus } + \ (4/3)\text{shear modulus}}{\rho}\right)^{\frac{1}{2}} \\
c_{s,S}(\text{shear}) &= \left(\frac{\text{shear modulus}}{\rho}\right)^{\frac{1}{2}} \\
c_{s,\text{rod}} &= \left(\frac{\text{Young's modulus}}{\rho}\right)^{\frac{1}{2}}
\end{aligned}
$$

For a fluid,

$$
c_{s,\text{fluid}} = \left(\frac{\text{bulk modulus}}{\rho}\right)^{\frac{1}{2}}
$$

Doppler Effect for Sound

Intrinsic and observed frequencies ν_e and ν_o respectively; emitter and receiver speeds relative to sound-carrying medium, V_e and V_s respectively, speed of sound, c_s.

$$
\begin{aligned}
\nu_o = \nu_e \left(\frac{c_s - V_r}{c_s + V_e}\right) &\quad \text{receiver and source getting further apart} \\
= \nu_e \left(\frac{c_s + V_r}{c_s - V_e}\right) &\quad \text{receiver and source coming closer together}
\end{aligned}
$$

Doppler Effect for light and photons in general

Defining θ as the angle between the relative motion of the emitter and receiver, and the line joining the two. $\beta \equiv (v/c)$, where v is the relative velocity between emitter and receiver (which are approaching if $\theta = 0$, receding if $\theta = \pi$),

$$
\begin{aligned}
\frac{\nu_o}{\nu_e} &= \frac{\sqrt{1-\beta^2}}{(1-\beta\cos\theta)}, \\
&= \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} \quad \text{(blueshift, direct approach)} \\
&= \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}} \quad \text{(redshift, direct recession)}
\end{aligned}
$$

Here, ν_e, and ν_o are the emitted and observed frequencies respectively. For energy, simply replace ν by E. For wavelength, use $\nu\lambda = c$ (i.e. just invert the expression for ν_o/ν_e). In the nonrelativistic limit ($\beta \ll 1$),

$$
\frac{\Delta E}{E_0} \sim \beta = \left(\frac{v}{c}\right)
$$

to first order in β.

Gravitational radius

$$\begin{aligned} r_g &= 1.4822 \left(\frac{M}{M_\odot}\right) \text{ km} \\ &= 1.4822 \times 10^{13}\ M_8 \text{ cm} \\ &\sim M_8\ \text{ AU} \end{aligned}$$

where $M_8 \equiv [M/(10^8 M_\odot)]$.

Light-crossing time

$$\begin{aligned} t &= r/c \\ &\sim 500 \left(\frac{r}{1\text{AU}}\right) \text{ seconds} \\ &\sim \left(\frac{r_g}{c}\right) \sim 500 M_8 \text{ seconds} \end{aligned}$$

where $M_8 \equiv [M/(10^8 M_\odot)]$.

2 Mechanics, Relativity, and Black Holes

Newton's second law

$$\mathbf{F} = m\frac{\mathrm{d}\mathbf{v}}{dt}$$

In one dimension we can also write $F = mv(\mathrm{d}v/\mathrm{d}r)$ using the chain rule, where $v \equiv |\boldsymbol{v}|$, and $v = (\mathrm{d}r/\mathrm{d}t)$.

One-dimensional Newtonian equations of motion with constant acceleration

$$\begin{aligned} v &= u + at \\ v^2 &= u^2 + 2ad \\ d &= ut + \frac{1}{2}at^2 \\ \text{Kinetic energy} &= \frac{1}{2}mv^2 = \frac{p^2}{2m} \\ \text{Work done} &= \int F\ \mathrm{d}r \\ \text{Power} &= Fv(r) \end{aligned}$$

In the above d and r are displacements, and the other symbols have their usual meanings.

Uniform circular motion

$$\begin{aligned}\text{Angular frequency} \quad \omega &= 2\pi\nu \\ &= \frac{2\pi}{T\ (\text{period})} \\ &= \frac{v\ (\text{tangential speed})}{r\ (\text{radius})} \\ F\ (\text{centrapetal force}) &= \frac{mv^2}{r}\end{aligned}$$

Center of mass

Two masses, m_1 and m_2 are separated by a distance d. The center of mass is located at a distance

$$\begin{aligned}x_1 &= \left(\frac{m_2}{m_1+m_2}\right) d \quad \text{from m}_1 \\ x_2 &= \left(\frac{m_1}{m_1+m_2}\right) d \quad \text{from m}_2\end{aligned}$$

Extended distribution

Taking moments about the origin of coordinates:

$$x_c = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i} \quad \text{discrete}$$

$$= \frac{\int_V x\ \rho(x,y,z)x\ \mathrm{d}\mathrm{d}y\mathrm{d}z}{\int_V \rho(x,y,z)\ \mathrm{d}x\ \mathrm{d}y\mathrm{d}z} \quad \text{continuous}$$

where ρ is the density.

Reduced mass

$$\mu \equiv \frac{m_1 m_2}{m_1+m_2}$$

Newton's law of gravity

$$\begin{aligned}F &= \frac{GMm}{r^2} \\ \text{Potential energy}, U &= -\frac{GMm}{r} \\ &= \left(\frac{r_g}{r}\right) mc^2\end{aligned}$$

where $r_g \equiv GM/c^2$ is the gravitational radius.

Escape velocity
From the surface of a spherical mass, M, with a radius r, the escape velocity is

$$v = \sqrt{\left(\frac{2GM}{r}\right)}, \quad \text{or} \quad \frac{v}{c} = \sqrt{\frac{2r_g}{r}}$$

where $r_g = 1.4822 \times 10^{13} M_8$ cm is the gravitational radius.

Surface gravity
At the surface of a spherical mass, M, with a radius r,

$$g = \frac{GM}{r^2},$$

which has units of acceleration.

Keplerian orbits
For simple circular orbits with a radius, r, of mass m orbiting a central mass, M ($m \ll M$), the orbital speed is

$$v = \sqrt{\left(\frac{GM}{r}\right)}, \quad \text{or} \quad \frac{v}{c} = \sqrt{\frac{r_g}{r}}$$

where $r_g = 1.4822 \times 10^{13} M_8$ cm is the gravitational radius.

More generally, for elliptical orbits the speed varies around the orbit:

$$v = \left[G(M+m)\left(\frac{2}{r} - \frac{1}{a}\right)\right]^{\frac{1}{2}}$$

where r is the separation of M and m, and a is the semimajor axis of the ellipse. If e is the eccentricity of the ellipse, the aphelion and and perihelion are given by

$$r_a \ = \ a(1-e), \quad r_p \ = \ a(1+e)$$

respectively, and the corresponding speeds can be found by substitution.

Continued on next page

Following are several convenient forms for the **orbital period**, T (for elliptical or circular orbits):

$$T = 2\pi \left[\frac{a^3}{G(M+m)}\right]^{\frac{1}{2}}$$

$$T(\text{years}) = \left[\frac{a_{\text{AU}}^3}{(M+m)(\text{solar masses})}\right]^{\frac{1}{2}}$$

$$T = 2\pi \left(\frac{r_g}{c}\right)\left(\frac{a}{r_g}\right)^{\frac{3}{2}}$$

$$T \sim 3100 \left(\frac{a}{r_g}\right)^{\frac{3}{2}} \left(\frac{M+m}{10^8 M_\odot}\right)^{-\frac{1}{2}} \quad \text{seconds}$$

$$T \sim 3.1 \times 10^{-5} \left(\frac{a}{r_g}\right)^{\frac{3}{2}} \left(\frac{M+m}{M_\odot}\right)^{-\frac{1}{2}} \quad \text{seconds}$$

where r_g is the gravitational radius. For circular orbits, a is equal to the radius.

For a general orbit, the total energy is

$$E \ = \ -\frac{GMm}{2a}$$

which is a constant of the motion, and the total angular momentum, L, is the other constant of motion.

$$\frac{L}{\mu} \ = \ \sqrt{[G(M+m)a(1-e^2)]}$$

where $\mu \equiv Mm/(M+m)$ is the reduced mass. Eliminating a in favor of E, we get

$$e^2 \ = \ 1 \ - \ \frac{L^2 E}{G^2\mu^2(M+m)Mm}.$$

Free-fall (Kelvin-Helmholtz) timescale

$$t_{\text{ff}} = \left(\frac{R^3}{GM}\right)^{\frac{1}{2}} \sim 1.59 \times 10^3 \ \left(\frac{M}{M_\odot}\right)^{-\frac{1}{2}} \left(\frac{R}{R_\odot}\right)^{\frac{3}{2}} \quad \text{seconds}$$

or about 26.6 minutes for the Sun.

Schwarzschild radius

$$r_s = \frac{2GM}{c^2} = 2r_g$$

Event horizon of a black hole

For a general rotating black hole, the event horizon, r_h in the equatorial plane is given by

$$\frac{r_h}{r_g} = 1 + \sqrt{(1-a^2)}$$

where $r_g \equiv GM/c^2$, and a is the dimensionless angular momentum of the black hole ($0 \leq a \leq 1$).

Ergosphere of a rotating black hole

$$r = r_g \left[1 + \left(1 - a^2 \cos^2 \theta\right)^{\frac{1}{2}}\right]$$

where θ is angle between the radius vector and the rotation axis.

Gravitational redshift in the field of a nonrotating black hole

If photons are emitted at a position r_e with an energy E_e in the field of a black hole, and an observer at a position r_o measures an energy E_o, then

$$\begin{aligned}
\frac{E_o}{E_e} &= \left[\frac{1 - \left(\frac{2r_g}{r_e}\right)}{1 - \left(\frac{2r_g}{r_o}\right)}\right]^{\frac{1}{2}} \\
&= \left[1 - \left(\frac{2r_g}{r_e}\right)\right]^{\frac{1}{2}} \quad \text{for an observer at } \infty \\
\frac{\Delta E}{E_o} \equiv \frac{(E_e - E_o)}{E_o} &\sim \frac{r_g}{r_e} \quad \text{at } \infty \text{ in the weak-field limit}
\end{aligned}$$

Lorentz transformations and special relativity

Defining

$$\beta \equiv \frac{v}{c}, \quad \frac{1}{\sqrt{1-\beta^2}},$$

the Lorentz transformations for relative motion in the x direction are

$$\begin{aligned}
t' &= \gamma \left[t - (xv/c^2)\right] \\
x' &= \gamma \left[x - vt\right] \\
y' &= y \\
z' &= z \\
E' &= \gamma m_0 c^2 \\
p' &= \gamma m_0 v
\end{aligned}$$

with the usual meanings of symbols. The energy-momentum relation is

$$E^2 = p^2c^2 + m_0^2c^4$$

and $E = m_0c^2$ for $v \ll c$. The kinetic energy is

$$\text{K. E.} = (\gamma - 1)m_0c^2.$$

If you are the observer, time intervals in the "other" frame are larger by a factor of γ, and so are lengths (in the direction of motion). It is necessary that time and length change in the same sense because c is constant in any frame. Lorentz *contraction* refers to the converse of this, that lengths in the other frame appear shorter in yours.

Relativistic addition of velocities
An object has a uniform velocity $\mathbf{u} = (u_x, u_y, u_z)$ with respect to a frame of reference that itself has a speed v along the x-axis of the observer's frame. The component u_x is in the same direction as v if u_x is positive. Now if the velocity of the object is $\mathbf{w}$ in the observer's frame, then the components of $\mathbf{w}$ in the observer's frame are:

$$w_x = \frac{u_x + v}{[1 + (u_x v/c^2)]} w_y \qquad = \frac{u_y}{\gamma[1 + (u_x v/c^2)]} w_z = \frac{u_z}{\gamma[1 + (u_x v/c^2)]}$$

To get the converse, i.e., if the observer switches reference frames, simply replace w with u on the L.H.S. of each equation, and u_x with $-w_x$ everywhere on the R.H.S. of each equation.

3 Radiation

Definition of specific intensity
Specific intensity, I_ν, is the energy carried by a ray bundle of radiation per unit area, per unit time, per unit frequency, per unit solid angle:

$$dE = I_\nu \, dA \, dt \, d\Omega \, d\nu$$

and I_ν/ν^3 is Lorentz invariant.

Flux

$$F_\nu = \int_\Omega I_\nu \cos\theta \, d\Omega = \int_0^\pi \int_0^{2\pi} I_\nu \cos\theta \sin\theta \, d\theta \, d\phi$$

(in spherical polar coordinates).

Pressure and energy density
Pressure, P_ν, and energy density, u_ν:

$$P_\nu = \frac{1}{c} \int_\Omega I_\nu \cos^2\theta \, d\Omega$$

$$u(\nu) = \frac{1}{c} \int_{\Omega} I_\nu \, d\Omega$$

$$P = \frac{1}{3} u \quad \text{for isotropic radiation}$$

Radiative transfer equation

$$\frac{dI_\nu}{d\tau} = -I_\nu + S_\nu$$

where τ is the optical depth to extinction along the beam, S_ν is the *source function* (the equation accounts for emission and scattering into the beam, and extinction along the beam). Formal solution is

$$I_\nu(\tau_\nu) = I_\nu(0) \exp(-\tau_\nu) \; + \; \int_0^{\tau_\nu} \exp[-(\tau_\nu - \tau_\nu')] \, S_\nu(\tau_\nu') \, d\tau_\nu'$$

but analytic solutions are possible only for the simplest cases; in general numerical techniques are required.

Kirchhoff's law of thermal radiation
For radiation and matter in equilibrium, the radiation spectrum depends only on temperature (T), and the source function is the Planck (blackbody) function:

$$S_\nu = B_\nu(T),$$

and I_ν approaches $B_\nu(T)$ at large optical depths.

Rosseland mean opacity formalism for plane-parallel geometry
One-dimensional problem in physical depth, z, temperature is a function of z, and $I_\nu = S_\nu = B_\nu$ in this approximation. If $\alpha = \alpha_a + \alpha_s$ is the total absorption plus scattering coefficient (units of reciprocal distance), a single number (α_R) corresponding to the extinction coefficient averaged over all frequencies, weighted by $\partial B / \partial T$ can be obtained:

$$\frac{1}{\alpha_R} \; = \; \frac{\int_0^\infty \frac{1}{\alpha_a + \alpha_s} \frac{\partial B_\nu}{\partial T} \, d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} \, d\nu} \; = \; \frac{\pi \int_0^\infty \frac{1}{\alpha_a + \alpha_s} \frac{\partial B_\nu}{\partial T} \, d\nu}{4 \sigma T^3}$$

The total flux, $F(z)$, integrated over frequency is then

$$F(z) \; = \; \frac{-16 \sigma T^3}{3 \alpha_R} \frac{\partial T}{\partial z}$$

where σ is the Stephan-Boltzmann constant.

Thermal bremsstrahlung
Also known as free-emission, or optically-thin thermal emission.

$$L_\nu = \bar{g}_{ff}\, 6.8 \times 10^{-38} Z^2 n_e n_i T^{-\frac{1}{2}} \exp\left[-h\nu/(kT)\right] \quad \mathrm{erg\ cm^{-3}\ s^{-1}\ Hz^{-1}}$$

and, integrated over all frequencies (bolometric),

$$L = 1.4 \times 10^{-27} T^{\frac{1}{2}} Z^2 n_e n_i \bar{g}_B (1 + 4.4 \times 10^{-10} T) \quad \mathrm{erg\ cm^{-3}\ s^{-1}}.$$

Here, $\nu =$ frequency;
$T =$ temperature of the plasma;
$Z =$ atomic number of the element making up the plasma;
$n_e =$ electron density;
$n_i =$ ion density;
$\bar{g}_{ff}$ velocity-averaged gaunt factor;
$\bar{g}_B$ frequency average of $\bar{g}_{ff}$ (range $\sim 1.1 - 1.5$).

The corresponding free-free absorption coefficient is:

$$\alpha_\nu^{ff} = 0.018 T^{-\frac{3}{2}} n_e n_i \nu^{-2} \bar{g}_{ff}$$

Gaunt factors in various regimes have been studied extensively in the literature in numerous works. See for example Karzas & Latter (1961, *Astrophysical Journal Supplement*, 6, 167), and discussion and references in *Radiative Processes in Astrophysics* by G. B. Rybicki & A. P. Lightman (1985: Wiley-VCH).

Stefan-Boltzmann law for Blackbody flux
For radiation in thermal equilibrium with matter at a temperature T, the flux, F, emitted per unit area is

$$F = \sigma T^4$$

where $\sigma = 5.67 \times 10^{-5}\ \mathrm{erg\ cm^{-2}\ deg^{-4}\ s^{-1}}$ is the Stefan-Boltzmann constant.

To get the energy density (u) instead of flux, replace σ by a, where $\sigma = (1/4)ac$:

$$u = aT^4$$

($a \sim 7.56 \times 10^{-15}\ \mathrm{erg\ cm^{-3}\ deg^{-4}}$.)

Blackbody spectrum or Planck function

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(h\nu/kT\right)-1} \quad \mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}\ steradian^{-1}}$$

Divide by c to get the energy density per unit steradian. If you want the total energy density for an isotropic field over all directions multiply by $4\pi/c$ instead. To convert the formulas to functions of wavelength, use $\nu\lambda = c$ and $\mathrm{d}\nu = -c\ \mathrm{d}\lambda/\lambda^2$. To convert to functions of energy use $E = h\nu$ and $\mathrm{d}E = h\mathrm{d}\nu$.

To get the Stefan-Boltzmann law integrate the spectrum over all frequencies, wavelengths, or energies. You will end up with an integral whose value is $\pi^4/15$ and this also yields the Stefan-Boltzmann constant in terms of fundamental constants:

$$\int_0^\infty \frac{x^3}{e^x-1}\mathrm{d}x \ = \ \frac{\pi^4}{15}, \quad \sigma = \frac{2k^4\pi^5}{15c^2h^3}$$

Wien displacement law

If $\nu_{\max}$, $\lambda_{\max}$ and $E_{\max}$ refer to the peak of the blackbody spectrum, the relation to the temperature (T) is the Wien displacement law. Various forms:

$$\begin{aligned}
\frac{\nu_{\max}}{T} &= 5.88\times10^{10}\ \mathrm{Hz\ deg^{-1}} \\
\lambda_{\max}T &= 0.290\ \mathrm{cm\ deg} \\
\lambda_{\max}T &= (5000\ \mathrm{Angstrom})(5800\ \mathrm{Kelvin}) \\
E_{\max} \equiv h\nu_{\max} &= 2.82kT(\mathrm{useful\ in\ the\ X-ray\ band})
\end{aligned}$$

Rayleigh-Jeans law for blackbody radiation

In the limit of low energies, low frequencies, and long wavelengths ($hv \ll kT$), the blackbody spectrum is:

$$\begin{aligned}
B_\nu &\sim \frac{2\nu^2kT}{c^2} \quad \mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}\ steradian^{-1}} \\
B_\lambda &\sim \frac{2ckT}{\lambda^4} \quad \mathrm{erg\ cm^{-2}\ s^{-1}\ cm^{-1}\ steradian^{-1}} \\
B_E &\sim \frac{2E^2kT}{c^2h^3} \quad \mathrm{erg\ cm^{-2}\ s^{-1}\ erg^{-1}\ steradian^{-1}}
\end{aligned}$$

Wien law spectrum for blackbody radiation

In the high-energy, high-frequency, short wavelength limit, ($hv \gg kT$), the blackbody spectrum is:

$$\begin{aligned}
B_\nu &\sim \frac{2h\nu^3kT}{c^2}e^{-h\nu/kT} \quad \mathrm{erg\ cm^{-2}\ s^{-1}\ Hz^{-1}\ steradian^{-1}} \\
B_\lambda &\sim \frac{2ckT}{\lambda^4}e^{-hc/\lambda kT} \quad \mathrm{erg\ cm^{-2}\ s^{-1}\ cm^{-1}\ steradian^{-1}} \\
B_E &\sim \frac{2E^2kT}{c^2h^3}e^{-E/kT} \quad \mathrm{erg\ cm^{-2}\ s^{-1}\ erg^{-1}\ steradian^{-1}}
\end{aligned}$$

Power radiated by an accelerated charge (Larmor's formula)
Nonrelativistic case, integrated over all emission angles:

$$P = \frac{2q^2a^2}{3c^3}$$

q = charge; a = acceleration. Differential power as a function of angle (Θ), also applies to an oscillating dipole:

$$\frac{dP}{d\Omega} = \frac{q^2a^2}{4\pi c^3} sin^2\Theta$$

Synchrotron radiation

$$\omega_B = \frac{qB}{\gamma mc}$$

Critical frequency:

$$\omega_c = \frac{3}{2}\gamma^3 \sin\alpha$$

where α is the angle between v and B. The power:

$$P(\omega) \propto \frac{q^3 B \sin\alpha}{mc^2} \times \text{[function of } (\omega/\omega_c)\text{]}$$

(see V. Ginzburg and S. Syrovatskii 1965, *Annual Reviews of Astronomy and Astrophysics*, 3, 297 for details). Total power:

$$P = \frac{4}{3}\sigma_T c(v/c)^2\gamma^2 U_B$$

where $U_B = B^2/8\pi$ is the magnetic field energy density.

Compton scattering
The energy/wavelength of a photon after scattering on an electron through an angle α in the electron rest frame,

$$\lambda = \lambda_0 + \frac{h}{m_e c}(1 - \cos\alpha)$$

(note $h/(m_e c) = \lambda_c$ = Compton wavelength). Here, m_e is the electron mass, and λ_0 and E_0 are the initial wavelength and energy respectively. The wavelength shift as a fraction of λ_c is

$$\frac{\Delta\lambda}{\lambda_c} = \frac{\lambda - \lambda_0}{\lambda_c} = (1 - \cos\alpha)$$

$$E = \frac{E_0}{1 + (E_0/m_e c^2)(1 - \cos\alpha)}.$$

where $E = hc/\lambda$, so $E/\lambda_c = E/m_e c^2$ = $E/(511 \text{ keV})$. Therefore, with E_0 and E in keV,

$$\frac{511 \text{ keV}}{E} = \frac{511 \text{ keV}}{E_0} + (1 - \cos\alpha)$$

The mean shift is 1 Compton wavelength (averaged over scattering angle) so

$$\frac{\overline{\Delta E}}{E_0} = \frac{z}{1+z}, \quad z \equiv \frac{E_0}{m_e c^2}$$

Note: $\lambda_c = 2.426310238 \times 10^{-12}$ m, or 0.024263$\mathring{A}$.

Differential Klein-Nishina cross-section for unpolarized incident radiation:

$$y_0 \equiv E_0/m_e c^2, \quad y \equiv E/m_e c^2 \quad w \equiv y_0/y \ = \ 1 + y_0(1 - \cos\alpha)$$

$$\begin{aligned}
\frac{\mathrm{d}\sigma_{\mathrm{KN}}}{\mathrm{d}\Omega} &= \frac{3\sigma_T}{16w^2}\left[w + \frac{1}{w} - \sin^2\alpha\right] \\
&= \frac{3\sigma_T}{16w^2}\left[w + \frac{1}{w} + \left(\frac{1}{y_0} - \frac{1}{y}\right)\left(2 + \frac{1}{y_0} - \frac{1}{y}\right)\right]
\end{aligned}$$

Many different ways to express the formula; last form does not use α explicitly.

Total Klien-Nishina cross-section integrated over scattering angle (from Martin V. Zombeck, in *Handbook of Space Astronomy and Astrophysics* 2006, Cambridge University Press) is (with $x \equiv E_0/m_e c^2$):

$$\sigma_{\mathrm{KN}} = \frac{3\sigma_T}{4}\left\{\frac{2(1+x)^2}{x^2(1+2x)} - \left[\frac{(1+x)}{x^3} - \frac{1}{2x}\right]\ln(1+2x) - \frac{(1+3x)}{(1+2x)^2}\right\}.$$

Eddington luminosity

$$\begin{aligned}
L_{\mathrm{Edd}} &= \frac{4\pi G M m_p c}{\sigma_T} \\
&= 1.263 \times 10^{38}\left(\frac{M}{M_\odot}\right) \quad \text{erg s}^{-1}
\end{aligned}$$

Electromagnetism and Maxwell's equations

Maxwell's Equations
The following versions of Maxwell's equations are in SI units.

Integral Form

$$\oint_S \mathbf{D} \cdot \mathbf{dS} = q$$
$$\oint_S \mathbf{B} \cdot \mathbf{dS} = 0$$
$$\oint_C \mathbf{E} \cdot \mathbf{dl} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{dS}$$
$$\oint_C \mathbf{H} \cdot \mathbf{dl} = I \ + \ \frac{d}{dt} \int_S \mathbf{D} \cdot \mathbf{dS}$$

Differential Form

$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

where

$$\mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E} \ + \mathbf{P}$$

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_r \mu_0} \ - \ \mathbf{M}$$

Lorentz Force

$$\mathbf{F} \ = \ q(\mathbf{E} \ + \ \mathbf{v} \times \mathbf{B})$$

Definitions of terms:

$\mathbf{S}$ and $\mathbf{dS}$ are surface area and element of area respectively.
$\int_S$ and $\oint_S$ denote integration over a surface and a closed surface respectively.
$\mathbf{D}$ = displacement current.
$\mathbf{P}$ = polarization field due to bound charges.

$\mathbf{E} \ = \ \nabla V$ electric field, where V is the electric potential.
$\mathbf{H} \ = \$ auxiliary magnetic field.
$\mathbf{M} \ = \$ magnetization of region/medium.
$\mathbf{B} \ = \$ magnetic field.
$q \ = \$ enclosed *free* charge.
$I \ = \$ current.
$J \ = \$ surface current density.
$\rho \ = \$ free charge density.
$\epsilon_0 \ = \$ permittivity of vacuum.
$\epsilon_r \ = \$ permittivity of medium relative to that of free space.
$\mu_0 \ = \ 1/(\epsilon_0 c^2) \ = \$ permeability of vacuum.
$\mu_r \ = \$ permeability of medium relative to vacuum.

Vector Potential Formulation
Maxwell's equations can be expressed in terms of a scalar and vector potential, V, and $\boldsymbol{A}$, respectively where

$$\boldsymbol{B} \equiv \boldsymbol{\nabla} \times \boldsymbol{A}$$
$$\boldsymbol{E} \equiv -\boldsymbol{\nabla} V - \frac{\partial \boldsymbol{A}}{\partial t}$$

However, since $\boldsymbol{A}$ does not uniquely determine $\boldsymbol{E}$ and $\boldsymbol{B}$, a supplemental condition called the *Lorentz gauge* is applied (the freedom is exploited to choose a condition that is allowed but convenient for other purposes such as deriving the wave equation):

$$\nabla \cdot \mathbf{A} \ = \ -\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$

Wave Equation
In free space without sources the wave equation for the electric field is:

$$\nabla^2 \mathbf{E} \ = \ \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \ = \ \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

The magnetic field, $\mathbf{B}$ satisfies an identical wave equation.

If we want to get a wave equation for the scalar and vector potentials (V and $\mathbf{A}$ respectively), we invoke the Lorentz gauge condition and get the following wave equations with sources:

$$\nabla^2 V \ - \ \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \frac{\rho}{\epsilon_0}$$
$$\boldsymbol{\nabla^2} \mathbf{A} \ - \ \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = \mu_0 \mathbf{J}$$

Note that Poisson's equation is simply the first of the above equations under steady-state conditions (time derivatives equal to zero).

Retarded Potentials

$$V \;=\; \int_{\mathcal{V}} \frac{[\rho] d\mathcal{V}}{4\pi\epsilon_0 r}$$

$$\mathbf{A} \;=\; \int_{\mathcal{V}} \frac{\mu_0 [\mathbf{J}] d\mathcal{V}}{4\pi r}$$

Energy Density and Poynting Flux

The Poynting vector, $\mathbf{N}$, is the rate of flow of energy per unit ares (flux):

$$\mathbf{N} = \mathbf{E} \times \mathbf{H}$$

which gives an outward energy flow rate over a closed surface of

$$P \;=\; \oint_S \mathbf{N} \cdot d\mathbf{S}$$

The energy density in the electromagnetic field is

$$u \;=\; \frac{1}{2}\mathbf{D} \cdot \mathbf{E} \;+\; \frac{1}{2}\mathbf{B} \cdot \mathbf{H}$$

4 Thermodynamics

Boltzmann factor
In a system at temperature T, the ratio of the numbers of "particles" occupying two energy levels, E_j and E_i is

$$\frac{N_j}{N_i} = \exp\left[-(E_j - E_i)/kT\right] = \exp\left[-\beta Eji\right]$$

where $\beta \equiv 1/(kT)$, and $E_j - E_i \equiv E_{ji} > 0$. The exponential is known as the Boltzmann factor.

Relation between Avogadro's constant, k, and R

$$R = N_A k,$$

reconciling kinetic theory with classical thermodynamics.

Equipartition
Each degree of freedom (d.o.f.) of a molecule in a gas at temperature T is associated with a mean

energy of $\frac{1}{2}kT$. There are 3 translational d.o.f. and between 0 and 3 rotational d.o.f., depending on the shape of the molecule. There may also be vibrational d.o.f. which may or may not be activated. Thus, for a monatomic gas each atom has a mean energy of $\frac{3}{2}kT$. For a diatomic gas with no vibrational d.o.f., the energy is $\frac{5}{2}kT$. More precisely, every term in the Hamiltonian appearing with a squared coordinate contributes a mean energy of $\frac{1}{2}kT$ per molecule.

Dalton's law of partial pressures

For a mixture of noninteracting gases, the partial pressures (p_k) due to each component add up to the total pressure of the gas (p). Conversely, the contribution of each component to the total pressure is proportional to the number of molecules.

$$P = \sum P_k, \quad P_k = x_k P$$

where x_k is the molar fraction of the kth component.

Ideal gas equation

$$\begin{aligned} PV &= NkT \\ P &= nkT \\ P &= \frac{\rho kT}{\mu m_{\mathrm{H}}} \\ PV &= N_A kT \quad \text{for 1 mole} \\ PV &= RT \quad \text{for 1 mole} \end{aligned}$$

where
$P =$ pressure,
$V =$ volume,
$N =$ total number of particles or molecules,
$n =$ number density of particles or molecules,
$\rho =$ mass density,
$m_{\mathrm{H}} =$ mass of Hydrogen atom,
$\mu =$ mean molecular weight,
$N_A =$ Avogadro's number,
$k =$ Boltzmann's constant,
$T =$ temperature in Kelvin.

$PV =$ constant is Boyle's law and V/T equal to a constant is Charles' law.

Heat capacities

Molar heat capacities:

$$\begin{aligned} C_V &= \left(\frac{đQ}{\mathrm{d}T}\right)_V \quad \text{constant volume} \\ C_p &= \left(\frac{đQ}{\mathrm{d}T}\right)_p \quad \text{constant pressure} \end{aligned}$$

First and Second laws of thermodynamics
First law for an ideal gas:

$$đQ = \mathrm{d}U + p\,\mathrm{d}V$$

Combined first and second law:

$$đQ = T\,\mathrm{d}S = \left(\frac{\partial U}{\partial T}\right)_V \mathrm{d}T + \left[\left(\frac{\partial U}{\partial V}\right)_T + p\right] \mathrm{d}V$$

For an ideal gas,

$$\left(\frac{\partial U}{\partial T}\right)_V = C_V, \quad \left(\frac{\partial U}{\partial V}\right)_T = 0,$$

the latter because internal energy does not depend on V. Thus

$$đQ \;\; = T\mathrm{d}S = \;\; C_V\,\mathrm{d}T + p\,\mathrm{d}V$$

so

$$C_p - C_V = p\left(\frac{\partial V}{\partial T}\right)_{\mathrm{p}} = R$$

for 1 mole, or $c_p - c_v = k$ per atom or molecule.

Maximum efficiency of a heat engine
A heat engine takes in heat Q_1 at a temperature T_1, puts out Q_2 at T_2, and does work $W = Q_1 - Q_2$. The requirement that the change in entropy, $(Q_2/T_2) - (Q_1/T_1) > 0$ means $(Q_2/Q_1) > (T_2/T_1)$ so $1 - (W/Q_1) > (T_2/T_1)$. Thus

$$\text{efficiency} \;=\; \frac{W}{Q_1} < \frac{T_1 - T_2}{T_1}$$

Polytropic, adiabatic, isothermal processes
For an ideal gas: define a generalized heat capacity, $c = \mathrm{d}Q/dT$, (not constant pressure, not constant volume). Define the polytropic index,

$$\gamma \equiv \frac{C_p - C}{C_V - C}, \quad \text{so} \quad PV^\gamma = \text{constant}, \quad TV^{\gamma-1} = \text{constant}$$

Equation of state can be expressed in the form of pressure proportional to some power of the mass density. Special cases:
Adiabatic, or equivalently, isentropic: $\gamma \equiv C_p/C_V$ ($c = 0$).
Isothermal: $\gamma = 1$. This is simply the ideal gas equation with constant T so PV = constant. (c is effectively infinite).
For a monatomic gas $(C_p/C_V) = 5/3 \sim 1.667$, for a diatomic gas with no vibrational degrees of freedom, $(C_p/C_V) = 7/5 = 1.4$.

Maxwell-Boltzmann velocity distribution for an ideal gas

$$f(v) = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{\frac{3}{2}} v^2 \exp\left(-mv^2/2kT\right).$$

where $m =$ mass of the particle, atom, or molecule in question. Distribution is only valid for nonrelativistic velocities. Above form is normalized to unity. The peak, $v_{\rm peak}$, and root-mean-square (r.m.s.), $v_{\rm r.m.s.}$, velocities are

$$v_{\rm peak} = \sqrt{\left(\frac{2kT}{m}\right)}, \quad v(\text{r.m.s.}) = \sqrt{\left(\frac{3kT}{m}\right)}$$

The following form is useful for an electron plasma with temperature known in keV:

$$v \sim 23,000 \left(\frac{kT}{1\ \rm keV}\right)^{\frac{1}{2}} \left(\frac{m_e}{m}\right)^{\frac{1}{2}} \ \ \rm km\ s^{-1}$$

Pressure in terms of momentum distribution of particles

$$P = \frac{1}{3}\int v\ p\ f(p) 4\pi p^2 {\rm d}p$$

Partition Function

Label a particular configuration of a system by r, composed of energy levels ϵ_i, each occupied by n_i particles.

$$Z = \sum_r \exp\left(-\beta E_r\right) \quad = \quad \sum_{\rm all\ n_i} \exp\left[-\beta \sum_{i=1}^{\infty} \epsilon_i n_i\right],$$

where $\beta \equiv 1/(kT)$. If the total number of particles or occupation of a given energy level is unconstrained by $N = \sum n_r$, Z simplifies to:

$$Z = \prod_{i=1}^{\infty} \sum_{n_i=1}^{\infty} \exp\left(-\beta \epsilon_i n_i\right)$$

The probability of finding the system in the state r is:

$$p_r(n_1\epsilon_1, n_2\epsilon_2 ...) = \frac{\exp\left(-\beta E_r\right)}{Z}$$

Partition function for a classical gas

The single-particle partition function, Z_1, is (neglecting internal energy states):

$$Z_1 = \int_0^{\infty} \frac{V 4\pi p^2\ {\rm d}p}{h^3} \exp\left(-\beta p^2/2m\right) = V \left(\frac{2\pi m kT}{h^2}\right)^{\frac{3}{2}}$$

which uses $\int_0^{\infty} x^2 e^{-\alpha x^2} = (\sqrt{\pi/\alpha})/(4\alpha)$. Then

$$Z = \frac{Z_1^N}{N!} = e^N \left(\frac{V}{N}\right)^N \left(\frac{2\pi m kT}{h^2}\right)^{\frac{3N}{2}}$$

which uses Stirling's approximation (abbreviated formula), $N! \sim [N/e]^N$.

Partition function for a quantum gas

$$\begin{aligned} Z_i &= \sum_{n_i=0}^{\infty} e^{-\beta(E_i-\mu)n_i} \\ &= 1 + e^{-\beta(E_i-\mu)}, \quad \text{Fermions, Fixed N},\ n_i = 0, 1 \\ &= [1 - e^{-\beta(E_i-\mu)}]^{-1}, \quad \text{Bosons, fixed N},\ \ n_i = 0 \text{ to infinity} \end{aligned}$$

The mean occupation number is

$$\begin{aligned} \bar{n}_i &= -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \mu} \\ &= \frac{1}{e^{\beta(E_i-\mu)}+1} \quad \text{for Fermions} \\ &= \frac{1}{e^{\beta(E_i-\mu)}-1} \quad \text{for Bosons} \end{aligned}$$

For photons, $\mu = 0$ because there is no restriction on the number of photons. In fact the mean occupation number for photons can be derived directly simply from

$$\begin{aligned} Z_i &= \sum_{n_i=0}^{\infty} \exp\left(-\beta E_i n_i\right) \\ &= \frac{1}{1-e^{-\beta E_i}} \\ \bar{n}_i &= -\frac{1}{\beta}\frac{\partial \ln Z}{\partial E_i} \\ &= \frac{1}{e^{\beta E_i}-1} \quad \text{for photons} \end{aligned}$$

Entropy

From the combined first and law of thermodynamics:

$$\Delta S \ = \ \int_{T_1}^{T_2} \frac{C_{\mathrm{v}}}{T}\, dT \ + \ \int_{V_1}^{V_2} \left(\frac{\partial p}{\partial T}\right)_{\mathrm{V}} dV$$

or

$$S \ = \ N[c_V \ \ln T + k \ln\left(\frac{V}{N}\right) + \ \text{constant}],$$

where S is the entropy. The above can be reconciled with statistical mechanics results (Sackur-Tetrode equation below). First:

$$
\begin{aligned}
S &\equiv k \ln \Omega \quad \text{(definition)} \\
&= k \sum_r p_r \ln p_r \\
&= \frac{U}{T} + k \ln Z
\end{aligned}
$$

For an ideal monatomic gas, using Z for a classical gas:

$$S = Nk \left\{ \frac{5}{2} + \frac{3}{2} \ln \left[\frac{V}{N} \left(\frac{2\pi mkT}{h^2} \right) \right] \right\}$$

(Sackur-Tetrode equation; not valid at T approaching zero).
For blackbody radiation in equilibrium with matter,

$$S = \frac{4aT^3V}{3}$$

Helmholtz function, internal energy, pressure
Helmholtz function: compare with derivation of entropy from statistical weights.

$$F = U - TS = -kT \ln Z$$

$$S = -\left(\frac{\partial F}{\partial T} \right)_{V,N}$$

Internal energy:

$$U = \frac{\partial \ln Z}{\partial \beta}$$

Pressure:

$$
\begin{aligned}
P &= \frac{1}{\beta} \left(\frac{\partial \ln Z}{\partial V} \right)_\beta \\
&= -\left(\frac{\partial F}{\partial V} \right)_T
\end{aligned}
$$

Density of States
Particles:

$$
\begin{aligned}
f(k)\mathrm{d}k &= \frac{Vk^2 \mathrm{d}k}{2\pi^2} \\
f(\omega)\mathrm{d}\omega &= \frac{V\omega^2 \mathrm{d}\omega}{2\pi^2 v^3} \\
f(p)\mathrm{d}p &= \frac{V 4\pi p^2 \mathrm{d}p}{h^3}
\end{aligned}
$$

where we use $k = 2\pi/\lambda$, $p = \hbar k$, $\omega = 2\pi\nu$, $\lambda = h/p$ (the latter is the de Broglie wavelength).

For photons we must account for two orthogonal polarizations, so multiply by 2 and replace v with c:

$$f(\omega)\mathrm{d}\omega = \frac{V\omega^2\mathrm{d}\omega}{\pi^2 c^3}$$
$$f(\nu)\mathrm{d}\nu = \frac{V 8\pi\nu^2\mathrm{d}\nu}{c^3}$$

Fermi energy

This is the upper-most energy occupied by a distribution of fermions in the limit of a temperature of zero. Simply "add up" the energies of each particle, taking into account the density of states.

$$E_F = \frac{h^2}{2m}\left(\frac{3N}{8\pi V}\right)^{\frac{2}{3}}$$

Bose-Einstein gas critical temperature

Bosons have integral spin and in a distribution of indistinguishable particles, when the chemical potential, $\mu = 0$, below a critical temperature, T_c (the Bose-Einstein condensation temperature), all the particles fall into the ground state. Obtain T_c by "adding up" all the partciles, accounting for theor occupation numbers and the density of states at T_c and $\mu = 0$:

$$\frac{N}{V} = \left(\frac{2\pi mkT_c}{h^2}\right)^{\frac{3}{2}}\left\{\frac{2}{\sqrt{\pi}}\int_0^\infty \frac{z^{1/2}\,\mathrm{d}z}{e^z - 1}\right\}$$

The integral has a value of 2.6124. Then

$$T_c \approx 3.32\frac{\hbar^2 n^{\frac{2}{3}}}{mk}$$

where n is the number density of particles. T_c can be of the order of nano-Kelvins.

5 Atomic physics

Classical Bohr radius

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_e e^2} = 5.291772108 \times 10^{-9}\ \mathrm{m}$$

where
$m_e =$ electron mass,
$e^2 =$ elementary charge,
$\hbar =$ Planck's constant divided by 2π.

Bohr magneton

$$\mu_B = \frac{e\hbar}{2m_e} \; = \; 9.2740 \times 10^{-24} \text{ J T}^{-1}$$

or 5.788×10^{-5} eV T^{-1}. (Values from NIST CODATA 2010.)

Schrödinger Equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi \; + \; V\psi \; = \; i\hbar\frac{\partial\psi}{\partial t} \; = \; E\psi$$

Hydrogen energy level spectrum
Both the semi-classical (Bohr) and the solution of Schrödinger's equation give the same result for the energy levels of a hydrogenic atom. The energy of the nth energy level is

$$\begin{aligned} E_n &= \; -\left(\frac{m_p m_e}{m_p + m_e}\right)\frac{Z^2 e^4}{8h^2\epsilon_0^2}\frac{1}{n^2} \\ &= \; -hcR_\infty\left(\frac{Z^2}{n^2}\right) \\ &= \; 13.60569253\left(\frac{Z^2}{n^2}\right) \; \text{ eV} \end{aligned}$$

where
$m_e =$ electron mass,
$m_p =$ proton mass,
$e =$ elementary charge,
$h =$ Planck's constant,
$c =$ speed of light,
$\epsilon_0 =$ permittivity of free space,
$Z =$ atomic number.
R_∞ is the Rydberg constant.

Sometimes the reduced mass, $m_p m_e/(m_p + m_e)$, is simply replaced by m_p. The error incurred in doing so is of the order of the same magnitude as the fine structure (i.e., $\sim 0.05\%$, or ~ 0.01 eV). The energy for transitions between different levels $n = q$ and $n = p$ (i.e., emission and absorption-line energies) is then

$$E_{pq} \; = \; 13.60569253 Z^2\left(\frac{1}{p^2} - \frac{1}{q^2}\right)$$

where $q > p$.

Hydrogen line spectrum

Transition	Series Name	Principal line	Continuum Limit
$1 \rightarrow n$	Lyman	$1216\mathring{A}$	$912\mathring{A}$
	Lyα, Lyβ ...	$2 \rightarrow 1$	Lyman limit
$2 \rightarrow n$	Balmer	$6563\mathring{A}$	$3648\mathring{A}$
	Hα, Hβ ...	$3 \rightarrow 2$	Balmer limit
$3 \rightarrow n$	Paschen	$18750\mathring{A}$	$8208\mathring{A}$
	Pα, Pβ ...	$4 \rightarrow 3$	Paschen limit

Fine structure energy levels in Hydrogen
To order $(v/c)^2$ the splitting of energy level n due to angular momentum and relativistic effects is

$$\Delta E_{n,j} \;=\; \frac{\alpha^2 m e^4 Z^4}{2\hbar^2(4\pi\epsilon_0)^2 n^4}\left(\frac{3}{4} - \frac{n}{j+\frac{1}{2}}\right)$$

$\alpha =$ fine structure constant,
$m =$ reduced mass $m_p m_e/(m_p + m_e)$,
$m_e =$ electron mass,
$m_p =$ proton mass,
$e =$ elementary charge,
$\hbar =$ Planck's constant divided by 2π,
$\epsilon_0 =$ permittivity of free space.

Note that there is also an effect that has no classical analog that is included. Also note that degeneracy in $\boldsymbol{l}$ is not lifted for H and H-like atoms, although it is lifted for $\boldsymbol{j}$.

Selection rule: $\Delta j = 0, \pm 1$. The difference in energy levels between different j values is proportional to n^{-3} so drops rapidly with n.

Fine structure constant

$$\alpha \;=\; \frac{e^2}{2\epsilon_0 hc} \;=\; 7.297 \times 10^{-3} \;\sim\; \frac{1}{137}$$

where
$e =$ elementary charge,
$h =$ Planck's constant.
$c =$ speed of light.
The fine structure constant is dimensionless. An interesting result is that the naïve classical orbital speed of an electron in a hydrogenic atom as a fraction of the speed of light is

$$\frac{v}{c} \sim \alpha\, Z$$

where Z is the atomic number of the nucleus. This gives an idea of whether relativistic effects are important for a given Z. Another result is that the binding energy of an electron in a hydrogenic atom can be expressed in terms of α and the electron rest mass, $m_e c^2$:

$$E \;=\; -\frac{1}{2}\alpha^2 Z^2 (m_e c^2).$$

The binding energy is equal to the ionization energy (or potential), or the energy required to remove the electron from the atom. Of course, α sets the scale for the corrections to the basic atomic energy levels due to relativistic effects and the interaction of the magnetic field due to the electron's orbital motion and the electron's own magnetic moment. This is known as the spin-orbit interaction. The resulting energy shifts with respect to the gross energy level structure are known collectively as the fine structure of the energy-level spectrum. The energy corrections are of the order of

$$\Delta E \sim \frac{1}{2}\alpha^2 Z^2\; E_{\text{gross}}$$

The classical electron radius, Compton wavelength, and Bohr radius are related by

$$r_e \;=\; \frac{\alpha \lambda_C}{2\pi} \;=\; \alpha^2 a_0,$$

where $\lambda_C = h/(m_e c)$ is the Compton wavelength.

Zeeman Effect

This refers to magnetic fields in the laboratory causing energy shifts comparable to the fine structure shifts. For fields of the order of 1 T, the energy shift is of the order of 10^{-4} eV. The levels are split into $2J+1$ levels, where J is the total orbital plus spin angular momentum quantum number.

The interaction energy is

$$\Delta E = g m_j \mu_B B$$

where m_j is an integer between $-J$ and J, corresponding to the quantum number of the z-component of the total angular momentum. The quantity g is

$$g = 1 \;+\; (g_e - 1)\frac{J(J+1) - L(L+1) + S(S+1)}{2J(J+1)}$$

where $g_e = 2.0023193$ (NIST 2010) is calculated including quantum electrodynamic effects. If these are not included, $g_e = 2$ gives what is known as the Landé formula. Note that the entire formalism is only valid for weak magnetic fields; if the field is strong enough to decouple $\mathbf{L}$ and $\mathbf{S}$, it breaks down (and it is known as the Paschen-Bach effect).

Spectroscopic term notation

Each level n is $2(2l+1)$-fold degenerate in electron energy (the extra 2 comes from the fact that there are two spin states of the electron). The total is

$$\sum_{0}^{n-1} 2(2l+1) \ = 2n^2$$

The degeneracy is accidental because of the inverse square form of the Coulomb law.

The letter notation for $\boldsymbol{l}$ and $\boldsymbol{L}$, where $\boldsymbol{L} = \boldsymbol{s} + \boldsymbol{l}$ is:

l:	s	p	d	f	g
L:	S	P	D	F	G
	0	1	2	3	4

General notation for a spectroscopic term is:

$$n \ {}^{2S+1}L_J$$

For example, for $2 \ {}^2P_{3/2}$, $n = 2$, $L = 1$, $J = 3/2$: $\boldsymbol{S}$ is aligned to be in the same direction as $\boldsymbol{L}$, so $J = 1 + \frac{1}{2} = \frac{3}{2}$. For $2 \ {}^2P_{1/2}$, $\boldsymbol{S}$ is aligned parallel to $\boldsymbol{L}$ but in the opposite direction. Then $J = 1 - \frac{1}{2} = \frac{1}{2}$.

Ordering of Energy Levels and Electronic Configuration

Shells are filled in order of increasing $n + l$ and for a given $n + l$, in order of increasing n. Examples of ground states (unexcited):

Ar (Z=18): $1s^2 2s^2 2p^6 3s^2 3p^6$
K (Z=19): $1s^2 2s^2 2p^6 3s^2 3p^6 4s^1$
Ca (Z=20): $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2$
Sc (Z=21): $1s^2 2s^2 2p^6 3s^2 3p^6 4s^2 3d^1$

Einstein Coefficients
Considering a two-level system and transitions between energy levels 1 and 2, which have degeneracies (multiplicities) g_1 and g_2 respectively, the rates for three processes are:

$A_{21} =$ transition rate for *spontaneous photon emission* due to electron dropping from level 2 to level 1.

$B_{12}\bar{J} =$ photon absorption causing electron to go from level 1 to level 2, where $\bar{J}$ is the local averaged radiation intensity at the energy corresponding to the energy difference between levels 1 and 2, or $E_2 - E_1$, or $h\nu_0 \equiv h\nu_2 - h\nu_1$.

$B_{21}\bar{J}$ = stimulated emission.

Balancing the rates in thermodynamical equilibrium leads to the following relations:

$$g_1 B_{12} = g_2 B_{21}$$
$$A_{21} = \frac{2h\nu_0}{c^2} B_{21}$$

Since the relations do not involve temperature, they must be true in general, even when thermodynamic equilibrium does not prevail. If we know any one of the three Einstein coefficients, we can compute the other two. The transition rate can be estimated for electric dipole transitions by approximating the matrix element by the electron charge multiplied by the Bohr radius as follows:

$$A_{21} = \frac{1}{4\pi\epsilon_0} \frac{4}{3} \frac{\nu_0^3}{\hbar c^3} |< 2|e\boldsymbol{r}|1 >|^2$$

or

$$A_{21} \sim \frac{\nu_0^3}{\hbar c^3} \frac{(ea_0)^2}{4\pi\epsilon_0} \nu_0^3 \sim \ 10^8 \ s^{-1}$$

The lifetime of an excited state against spontaneous decay is $\sim 10^{-8}$ s. The classical value of the transition probability is equal to the classical radiation damping constant, given by

$$\gamma_{\text{classical}} = \frac{1}{4\pi\epsilon_0} \left(\frac{8\pi^2 e^2}{3m_e c^3} \right) \nu_0^2 = 2.47 \times 10^{-22} \ \nu_0^2 \ \text{s}^{-1}$$

The damping constant is also an indicator of the natural width of a transition line and the profile is a Lorentzian, and the FWHM is $\gamma/2\pi$:

$$I(\nu) = I_0 \frac{\gamma/4\pi^2}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2}$$

where γ is the damping width. The oscillator strength, f_{nm}, is defined by relating the classical transition rate to the actual quantum-mechanical transition rate so that for a transition from level n to level m

$$A_{nm} \equiv 3\frac{g_m}{g_n} f_{nm}\gamma = \frac{g_m}{g_n} \frac{1}{4\pi\epsilon_0} \frac{8\pi^2 e^2 \nu^2}{m_e c^3} f_{nm}$$

or

$$B_{nm} = \frac{1}{4\pi\epsilon_0} \frac{\pi e^2}{m_e h\nu} f_{nm}.$$

The oscillator strength is dimensionless and represents the fraction of line emission going into a particular channel. For hydrogen, a semi-analytic form can be derived:

$$f_{nm} = \frac{64}{3\sqrt{3}\pi} \frac{1}{g_m} \left(\frac{1}{m^2} - \frac{1}{n^2} \right)^{-3} \frac{g_G}{n^3 m^3}$$

where g_G is the Gaunt factor for bound-bound transitions (e.g., see Karzas & Latter 1961, ApJS, 6, 167). The above formula is called Kramer's formula. In terms of the oscillator strength, the total absorption cross-section from the ground state (integrated over energy or frequency) is Ladenburg's relation:

$$\sigma_{\rm abs} \;=\; \frac{1}{4\pi\epsilon_0}\frac{\pi e^2}{m_e c} f_{nm} \;=\; 0.0265 f_{nm}\;{\rm cm}^2\;{\rm Hz}$$

Over the profile of the line, if we don't integrate over energy, we get

$$\sigma_{{\rm abs},\nu} \;=\; \frac{1}{4\pi\epsilon_0}\frac{\pi e^2}{m_e c} f_{nm}\,\frac{\gamma/4\pi^2}{(\nu-\nu_0)^2+(\gamma/4\pi)^2}$$

where γ is the radiation damping constant from above. For natural lines (i.e., no broadening aside from natural broadening), the absorption cross-section is simply

$$\sigma_{{\rm abs},\lambda_0} \;=\; \frac{9}{2\pi}\lambda_0^2$$

where $\nu_0\lambda_0 = c$. This comes from evaluating $\sigma_{{\rm abs},\nu}$ at the line center ($\nu=\nu_0$) and using the expression for the classical damping constant and the relation $f_{nm}/\gamma = 3/\gamma_{\rm classical}$, following from the preceeding relations.

21 cm Hydrogen line

The HI 21cm line is due to the hyperfine structure splitting of the $1s^2S_{1/2}$ ground level. The energy difference corresponds to 5.8×10^{-6} eV, or 1420 Hz, or 21.11 cm. The transition is forbidden and has a very low rate (1 every 10 million years) but there is so much hydrogen in the Universe that the line is commonly observed.

Helium-like triplet diagnostics

Resonance line (w or r): ^{1}P to ^{1}S; intercombination lines ($x+y$, or i): ^{3}P to ^{1}S; forbidden line (z or f): ^{3}S to ^{1}S.

$$R \equiv \frac{f}{i} \quad \text{is sensitive to the gas density}$$

$$G \equiv \frac{i+f}{r} \quad \text{is sensitive to the gas temperature}$$

Equivalent width of absorption or emission lines

If the total flux in a spectrum at frequency ν is $F_{{\rm net},\nu}$ (say photons ${\rm cm}^{-2}\;{\rm s}^{-1}\;{\rm Hz}^{-1}$), and the flux in the continuum only is $F_{c,\nu}$, the equivalent width (W_{ν_0}) of an absorption or emission line centered at ν_0 (or wavelength λ_0) is the integral of the ratio of the flux in the line (which could be negative) to the flux in the continuum.

$$W_{\nu_0} = \int_0^\infty \left(1-\frac{F_{{\rm net},\nu}}{F_{c,\nu}}\right)\,{\rm d}\nu$$

In terms of wavelength:

$$W(\mathring{A}) = \frac{\lambda_0^2}{c}W_{\nu_0}$$

In the case that the line is not spectrally resolved, W_{ν_0} is simply the width of the continuum around the line that contains the same flux as the line:

$$W_{\nu_0} = \frac{\text{total line flux}}{\text{continuum flux at line center}} = \frac{F(\text{line})}{F_{c,\nu}}$$

The equivalent width has units of frequency, energy or wavelength. Therefore, it transforms relativistically and cosmologically like other quantities with the same units. For example, W in units of energy transforms with cosmological redshift as

$$W(\text{energy units, frequency})_{\text{observed}} = \frac{W(\text{energy units, frequency})_{\text{rest-frame}}}{(1+z)}$$
$$W(\text{wavelength units})_{\text{observed}} = (1+z)W(\text{wavelength})_{\text{rest-frame}}$$

Curve of growth

In a *Curve of growth*, W_λ/λ_{ij} is plotted against $N\lambda_{ij}f_{ij}$, where a column density, N, of particular ions produces an absorption-line equivalent width of W_λ for transitions between energy levels i and j, with an oscillator strength of f_{ij} . (Equivalently, W_λ is plotted against the optical depth at the line center, which is $\propto N$.) Objective: to deduce the column density, N, of the ion responsible for the absorption line, by relating N to measureables of the line, such as W_λ.

If the line is spectrally resolved:

$$N = \frac{1}{\sigma_L}\int \tau_\nu \, \mathrm{d}\nu = \frac{4\epsilon_0 m_e c}{e^2}\frac{c}{f_{ij}\lambda_{ij}^2}\int \tau_\lambda \, \mathrm{d}\lambda = 1.13 \times 10^{20} \text{ cm}^{-2}\frac{1}{f_{ij}[\lambda_{ij}(\text{Å})]^2}\int \ln\left(\frac{F_{\text{net},\lambda}}{F_{c,\lambda}}\right) \mathrm{d}\lambda$$

If the line is spectrally unresolved

Three cases: (a) $\tau_\nu \ll 1$, (b) $\tau_\nu \sim 1$ to a few, (c) $\tau_\nu \gg 1$. For (a) use formula below, for (b) and (c) plot curves of growth (W_λ/λ_{ij} versus $N\lambda_{ij}f_{ij}$, overlay data to deduce N).

$$N = 1.13 \times 10^{20} \text{ cm}^{-2} \frac{1}{f_{ij}[\lambda_{ij}(\text{Å})]^2}W_\lambda \quad (\tau_0 \ll 1); \qquad W_\lambda \propto \tau_0$$
$$\frac{W_\lambda}{\lambda_{ij}} = \frac{2DF(\tau_0)}{c}, \quad \tau_0 = \frac{0.0150N\lambda_{ij}f_{ij}}{D}, \quad (\tau_0 \sim 1); \qquad W_\lambda \propto (\ln \tau_0)^{\frac{1}{2}}$$
$$\frac{W_\lambda}{\lambda_{ij}} = \frac{2\lambda_{ij}}{c}\left[\frac{Nf_{ij}e^2\gamma}{4\epsilon_0 m_e c}\right]^{\frac{1}{2}} \quad (\tau_0 \gg 1); \qquad W_\lambda \propto (\tau_0)^{\frac{1}{2}}$$

In the last equation, γ is the radiation damping constant.

Emission measure and free-free optical depth

$$\text{Emission Measure, E.M.} \equiv \int N_e^2 \, \mathrm{d}s$$

If s is in parsecs, N_e in cm^{-3}, E.M. is clearly in $\mathrm{cm}^{-6}\ \mathrm{pc}^{-1}$. A useful expression for the free-free optical depth of a hot plasma (due to Altenhoff, W., Mezger, P. G., Wendker, H., & Westerhout, G.

1960, Veröff, Sternwarte, Bonn, No. 59, 48) is

$$\tau_{\nu,\mathrm{ff}} \; = \; 8.235 \times 10^{-2} T_e^{-1.35} \nu^{-2.1} \int N_e^2 \mathrm{d}s$$

valid for $\nu \ll 10^{10} T_e$, and $T_e < 9 \times 10^5$ K. Here, T_e is the electron temperature in Kelvin, ν is the frequency in GHz, and the emission measure is in units of $\mathrm{cm}^{-6}\ \mathrm{pc}^{-1}$.

Saha equation

The Saha equation describes the balance of the number densities of ions in adjacent ionization states (say, i and $i+1$), in a hot plasma that is in thermal equilibrium.

$$\frac{n_{i+1} n_e}{n_i} = \frac{2 Z_{i+1}(T)}{Z_i} \left(\frac{2\pi m_e k T}{h^2} \right)^{\frac{3}{2}} \exp\left(-E_{i,i+1}/kT\right)$$

where n_i, n_{i+1}, n_e are the number densities of the relevant quantities, and $E_{i,i+1}$ is the energy required to completely remove an electron (ionize) from an ion with ionization state i taking it to ionization state $i+1$. Both ions are in their ground state. Z_i and Z_{i+1} are the partition functions of the ions labeled i and $i+1$ respectively, and these are sums of the degeneracies (multiplicities, or statistical weights) over all energy levels, weighted by the Boltzmann factor for the difference between each energy level and ground state energy level:

$$Z_i = \sum_{j=1}^{\infty} g_{i,j} \exp\left[-(E_j - E_1)/kT\right]$$

Note that Z includes internal energy states as well as including degeneracies in internal states. Thus, for neutral hydrogen, $g_1 = 4$ (2 spin states of the proton and 2 spin states of the electron). For ionized hydrogen, the degeneracy is just 2. If $E \ll kT$, the Boltzmann factor may be neglected, in which case the partition functions reduce simply to degeneracies.

Use $P_e = n_e kT$ to replace n_e with electron pressure.

Ionization parameter for photoionized gas
Two most commonly used:

$$U = \frac{Q}{4\pi r^2 n_e c} = \frac{1}{4\pi r^2 n_e c} \int_{1\ \mathrm{Rydberg}}^{\infty} N(E)\,\mathrm{d}E$$

or

$$\xi = \frac{L}{r^2 n_e} = \frac{1}{r^2 n_e} \int_{1\ \mathrm{Rydberg}}^{\infty} E N(E)\,\mathrm{d}E$$

where
$Q =$ number of photons in the continuum above 1 Rydberg, or 13.6 eV (i.e., the hydrogen ionization energy).
$L =$ luminosity in the continuum above 1 Rydberg.
$n_e =$ electron density of the plasma that is being photoionized.
$r =$ distance between the ionizing radiation source and the plasma.
$c =$ speed of light.
$N(E) =$ photon spectrum (e.g. photons s^{-1} keV^{-1}).

6 Stars, and Stellar Structure

Gravitational binding energy of a sphere

$$U = \frac{3GM^2}{5R}$$

Pressure at the center of a gravitationally bound sphere
On dimensional grounds:

$$P_c \sim \frac{M^2 G}{R^4}$$

Actual value at the center of the Sun and stars in general must be deduced from full models. For the Sun, $P_c \sim 2.5 \times 10^{16}\ \mathrm{N\ m^{-2}}$ (see http://nssdc.gsfc.nasa.gov/planetary/factsheet/sunfact.html).

Polytropic equation of state for gas and radiation
Parameterize the gas pressure as a fraction of the total gas plus radiation pressure with β, and eliminate T in the gas and radiation pressure formulas:

$$P_{\mathrm{gas}} = \frac{\rho k T}{\mu m_{\mathrm{H}}} \equiv \beta p_{\mathrm{total}}$$

$$P_{\mathrm{total}} = \left(\frac{k}{\mu m_H}\right)^{\frac{4}{3}} \left[\frac{3(1-\beta)}{a\beta^4}\right]^{\frac{1}{3}} \rho^{\frac{4}{3}}$$

Here μ is the mean molecular weight.

Pressure scale height

$$
\begin{aligned}
P &= P_0 \exp\left[-(\mu g m_{\mathrm{H}} h)/kT\right] \equiv P_0 e^{-h/H} \\
H &\equiv \frac{kT}{\mu g m_{\mathrm{H}}} \qquad \text{(isothermal)} \\
H &\equiv -\frac{\mathrm{d}h}{\mathrm{d}\ln P} \qquad \text{(even if not isothermal)}
\end{aligned}
$$

Equations of stellar structure
Hydrostatic equilibrium

$$\frac{\mathrm{d}P}{dr} = -\frac{Gm(r)\rho(r)}{r^2}$$

Mass continuity equation

$$m(r) = \int_0^r 4\pi r'^2 \rho(r')\, \mathrm{d}r'$$

or

$$\frac{\mathrm{d}m}{\mathrm{d}r} = 4\pi r^2 \rho$$

Radiative equilibrium

$$\frac{\mathrm{d}L}{\mathrm{d}r} = 4\pi r^2 \rho L_{\mathrm{gen}}$$

where L_{gen} is the energy generation rate.
Radiative transport

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{-3\bar{\kappa}\rho L(r)}{16\pi a c r^2 T^3}$$

Equation of state

$$
\begin{aligned}
p &= \frac{\rho k T}{\mu m_{\mathrm{H}}}, \qquad \text{gas pressure} \\
&= \frac{1}{3} a T^4, \qquad \text{radiation pressure}
\end{aligned}
$$

Energy generation rate

$$L_{\mathrm{gen}} = L_{\mathrm{gen},0}\ \rho T^n \qquad \text{(just a possible parmeterization); } n \text{ depends on mechanism}$$

Opacity

$$\kappa = \kappa_0 \rho T^{-7/2}, \qquad \text{(choice of Kramer's law)}$$

The resulting R, M, T, and L relations, assuming either gas-pressure domination and the pp chain (appropriate for low-mass stars), or radiation pressure domination and the CNO cycle (appropriate for

high-mass stars) are:
gas-pressure dominated:

$$R \propto M^{0.2}, \quad L \propto T_s^{6.75} T_s \propto M^{0.8}, \quad L \propto M^{5.4}$$

radiation-pressure dominated:

$$R \propto M^{0.45}, \quad L \propto T_s^{31.25} T_s \propto M^{0.049}, \quad L \propto M^{1.52}$$

Here, T_s is the surface temperature of the star. Actual relations depend on many details, including a star's chemical composition, and should be obtained from full-up numerical stellar structure models. Observed L versus M relation is $L \propto M^4$ (very roughly).

Note that plotting absolute magnitude versus B-V on an H-R diagram is equivalent to plotting L against T_s.

Convective Instability

$$\begin{aligned}
\left|\frac{\mathrm{d}T}{\mathrm{d}r}\right| &> \left|\frac{\mathrm{d}T}{\mathrm{d}r}\right|_{\text{adiabatic}} \\
\left|\frac{\mathrm{d}\ln T}{\mathrm{d}r}\right| &> \left|\frac{\mathrm{d}\ln T}{\mathrm{d}r}\right|_{\text{adiabatic}} \\
\left|\frac{\mathrm{d}\ln P}{\mathrm{d}\ln T}\right| &< \left|\frac{\mathrm{d}\ln P}{\mathrm{d}\ln T}\right|_{\text{adiabatic}}
\end{aligned}$$

Using $p/T^{\gamma/(1-\gamma)} = \text{constant}$,

$$\left|\frac{\mathrm{d}\ln P}{\mathrm{d}\ln T}\right| < \frac{\gamma}{\gamma - 1}$$

Eddington approximation
Impose the boundary condition at the surface that the flux is half of what it should be for the blackbody temperature. In addition, a grey atmosphere is assumed.

$$\begin{aligned}
\text{Flux} &= c\frac{\mathrm{d}p}{\mathrm{d}\tau} \\
&= \sigma T_s^4 \\
p &= \frac{\sigma}{c}\int T_s^4 \, \mathrm{d}\tau \\
&= \frac{\sigma}{c} T_s^4 \left(\tau + \frac{2}{3}\right) \\
&= \frac{1}{3} a T^4 \\
T^4 &= \frac{3}{4} T_s^4 \left(\tau + \frac{2}{3}\right)
\end{aligned}$$

where we have used $\sigma = ac/4$ ($sigma$ is the Stefan-Boltzmann constant and a is the radiation constant).

Limb darkening
Start with the solution to the radiative transport equation in a plane-parallel atmosphere, with an expression for the intensity at the surface:

$$I_\nu(\mu, \tau_\nu = 0) \;=\; \frac{1}{\mu}\int_0^\infty S(\tau_\nu)\exp\left(-\tau_\nu/\mu\right)\,\mathrm{d}\tau_\nu$$

Assuming a grey atmosphere (no frequency dependence of optical depth):

$$\frac{I_\nu(\theta)}{I_\nu(0)} = \frac{a+b\mu}{a+b} = (\frac{2}{5}+\frac{3}{5}\mu) = 0.4 + 0.6\cos\theta$$

Sedov-Taylor expansion of a supernova

$$R \sim 14\left(\frac{E_{51}}{n}\right)^{\frac{1}{5}}\left(\frac{t}{10^4\ \mathrm{years}}\right)^{\frac{2}{5}} \quad \mathrm{parsecs}$$

where E_{51} is the initial energy in units of $10^{51}\ \mathrm{erg\ s^{-1}}$, n is the number density of particles, in cm^{-3}, and t is in years. The expression assumes a monatomic gas ($\gamma = 5/3$). The temperature, T, is given by:

$$T \sim \left(\frac{E_{51}}{n}\right)R^{-3} \quad \mathrm{Kelvin}$$

where R is in parsecs.

7 Degenerate Matter

Equations of state
Pressure, P, mass density, ρ:

$$P \propto \rho^{\frac{4}{3}}, \qquad \text{(nonrelativistic)}$$
$$P \propto \rho^{\frac{5}{3}}, \qquad \text{(relativistic)}$$

The combined hydrostatic equilibrium and mass continuity equations give:

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = -4\pi G\rho$$

Assuming a polytropic equation of state, $P = K\rho^{[1+(1/n)]}$, rewrite with new variables:

$$\rho = \rho_c x^n$$
$$P = K\rho_c^{\frac{n+1}{n}} x^{n+1}$$
$$a \equiv \left[\frac{(n+1)K\rho_c^{\frac{1-n}{n}}}{4\pi G}\right]^{\frac{1}{2}}$$
$$r = az$$

to get

$$\frac{1}{z^2}\frac{\mathrm{d}}{\mathrm{d}z}\left(z^2\frac{\mathrm{d}x}{\mathrm{d}z}\right) = -x^n$$

This is the Lane-Emden equation, which in general must be solved numerically.

Mean molecular electron mass

The mean electron molecular mass, μ_e, converts number density of free electrons, n_e, to a mass density, ρ, and is defined by the equation $m_{\mathrm{H}}\mu_e n_e \equiv \rho$. The value of μ_e depends on the ionization state and relative element abundances and it measures (approximately) the number of nucleons per electron. If X is the mass fraction of hydrogen,

$$\mu_e = \frac{2}{1+X}$$

assuming that all elements heavier than hydrogen, each electron is associated with two nucleons. In white dwarfs, the hydrogen is supposed to be all depleted, in which case $X = 0$, $\mu_e = 2$.

Chandrasekhar mass limit for white dwarfs and mass-radius relation

As the mass of a white dwarf increases, the matter becomes relativistic and consideration of the total energy (kinetic plus gravitational) shows that there is no value of mass and radius that can prevent collapse. Numerical solution of the Lane-Emden equation shows that the critical (or Chandrasekhar) mass is

$$M_{\mathrm{crit}} = \frac{5.80}{\mu_e^2}\, M_\odot$$
$$= 1.45\, M_\odot$$

where the latter follows from the fact that in white dwarfs, the hydrogen is supposed to be all depleted, in which case $X = 0$, $\mu_e = 2$.

Below the critical mass, the equations show that $R \propto M^{-3}$.

8 Accretion and gravitational effects

Accretion efficiency
For accretion onto black holes there is no hard surface and there are many uncertainties in the accretion process. It is customary to parameterize the uncertainties in terms of an efficiency with which rest mass is converted to radiation. The radius is conventionally taken as the Schwarzschild radius ($= 2GM/c^2$) for the purpose of the definition:

$$L = \frac{2\eta GM\dot{M}}{R} = \eta\dot{M}c^2$$

The maximum efficiency for a non-rotating (Schwarzschild) black hole is $\sim 5.7\%$, and that for a maximally-rotating Kerr black hole is more than 30%; the actual value is debated. Compare with the efficiency of 0.7% for the hydrogen-to-helium nuclear burning pp-chain.

Accretion disk temperature and viscous dissipation
Dynamical viscosity relates the shear stress to the velocity gradient. Kinematic viscosity, ν_k, is the dynamic viscosity divided by the density. The dissipation rate per unit disk surface area, due to kinematic viscosity, in a simple geometrically thin disk at radius R from the center, $E(R)$, due to torque between adjacent rings rotating at the Keplerian velocity, ω_K, is,

$$E(R) = \frac{1}{2}\nu_k \rho H \left[R\frac{\mathrm{d}\omega_K}{\mathrm{d}R}\right]^2 = \frac{9}{8}\nu_k\rho H\frac{GM}{R^3}$$

where H is the disk height, and

$$\omega_K = \left(\frac{GM}{R^3}\right)^{\frac{1}{2}}$$

In the *alpha prescription*, or *alpha viscosity*, of disk models, the viscosity is simply parameterized by

$$\nu_k \equiv \alpha c_s H,$$

where the unknown quantity is simply expressed in a different way (c_s is the sound speed). Imposing the condition of steady radial angular momentum flow and Keplerian velocities right up to the accretion radius, R_*, the radial flow velocity can be expressed in terms of $\dot{M}$, giving after some manipulation,

$$\nu_k\rho H = \frac{\dot{M}}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right]$$

so that

$$E(R) = \frac{3GM\dot{M}}{8\pi R^3}\left[1 - \left(\frac{R_*}{R}\right)^{\frac{1}{2}}\right]$$

Note that the luminosity from the binding energy (K.E. plus P.E.) is half of $GM\dot{M}/2R_*$, but the accretion luminosity is defined as $GM\dot{M}/R$, which is the P.E. at the accretion surface. In other words, not all of it is released into radiation, only half. Formally, this can also be obtained from integrating the dissipation equation:

$$L_{\rm disk} = 2\int_{R_1}^{R_2} E(R) 2\pi R \, \mathrm{d}R$$

where the factor 2 accounts for there being two radiating faces of the disk.

If the local emission is a blackbody, $\sigma T^4 = E(R)$, so the disk temperature, T, is:

$$T(R) = \left\{ \frac{3GM\dot{M}}{8\pi R^3 \sigma} \left[1 - \left(\frac{R_*}{R} \right)^{\frac{1}{2}} \right] \right\}^{\frac{1}{4}}$$

where σ is the Stefan-Boltzmann constant.

Roche Limit

This limit corresponds to the closest distance of approach, d, of a mass m orbiting a much larger mass M, before the smaller mass is broken by tidal disruption. A simplistic approach for a solid body gives

$$d \sim R \left(\frac{2\rho_M}{\rho_m} \right)^{\frac{1}{3}},$$

and for a fluid body gives

$$d \sim 2.44 R \left(\frac{\rho_M}{\rho_m} \right)^{\frac{1}{3}}$$

where ρ_m and ρ_M are the densities of the smaller and larger bodies respectively, and R is the radius of the larger mass.

See Aggarwal and Oberbeck (1974, Astrophysical Journal, 191, 577) and references therein for more detailed calculations, which actually disagree with Roche due to different assumptions about elasticity (they get 1.957 instead of 2.44).

Hill Radius

If a test mass is introduced into a two-body gravitationally-bound system with masses m and M ($m \ll M$), whether the test mass is dominated by the gravity of the mass m or M depends on the position of the test mass in relation to the Hill spheres of the two masses m and M. If the orbit of the mass m is characterized by a semimajor axis a, and an eccentricity, e, an approximation for the Hill radius of the mass m is:

$$r_{{\rm H},m} \approx a(1-e) \left(\frac{m}{3M} \right)^{\frac{1}{3}} = a(1-e) \frac{R_m}{R_M} \left(\frac{\rho_m}{3\rho_M} \right)^{\frac{1}{3}}$$

where ρ and R are the density and radius respectively, of the appropriate masses. This approximation, from Hamilton and Burns (1992; Icarus, 96, 43) is commonly adopted.

9 Galaxies

De Vaucouleur's Law
The De Vaucouleur's law is an empirical relationship between the observed intensity or flux per unit area (i.e., surface brightness) of an elliptical galaxy's two-dimensional image on the sky, as a function of the radial distance, r, from the center of the galaxy. By convention, the relation is normalized at the radius which encloses half of the total luminosity (that radius is labeled r_e).

$$I(r) = I(r_e) \exp\left\{-7.669\left[\left(\frac{r}{r_e}\right)^{\frac{1}{4}} - 1\right]\right\}$$

(A generalized form of this law where the exponent is $1/n$ instead of $1/4$ is known as Sersic's law.) The units of flux may be, for example, $\mathrm{erg\ arcminute}^{-2}$. It is possible to use magnitudes per unit area as well, in which case the logarithmic form of de Vaucouleur's law has to be used.

For inferring actual three-dimensional runs of physical parameters with radius by deprojection, see, for example, Mazure & Capelato (2001), http://arxiv.org/abs/astro-ph/0112147. Note that for spiral galaxies, the fall of surface brightness with radius is simply exponential.

Faber-Jackson relation
The Faber-Jackson relation is an empirical "law" connecting the intrinsic luminosities of elliptical galaxies, L, with their velocity dispersions, σ_0. (The velocity dispersion is a measure the width of the distributions of stellar velocities in excess of systematic motion in the galaxy due to, e.g., rotation.)

$$L \propto \sigma_0^x$$

Typical velocity dispersions are of the order of hundreds of $\mathrm{km\ s}^{-1}$. The exact value of x depends on many factors including the way the galaxy sample is selected, and the factors which affect x are still being debated (e.g., see Nigoche-Netro et al. (2011, http://arxiv.org/abs/astro-ph/1107.6017); Forcadi & Malasavi (2012, http://arxiv.org/abs/1207.2750). The latter authors obtain x in the range of 3.4 to 5.6, depending on assumptions and selection crietria.

Tully-Fisher relation
The Tully-Fisher relation is an empirical "law" that connects the intrinsic luminosity of a spiral galaxy with the maximum rotational velocity (which occurs at some distance from the center of the galaxy). In general,

$$L \propto V_{\max}^y$$

where y depends on the sample selection and other factors (which are still the subject of research). Originally, Tully & Fisher reported $y = 2.5 \pm 0.3$ based on so-called photographic magnitudes (weighted towards the blue visible region). Many studies and values of y can be found in the literature. A recent one by Torres-Flores et al. (2010; http://arxiv.org/abs/1003.0345) investigating the B-band relation found $y = 2.94$.

Virial radius for clusters of galaxies
A cluster of galaxies is said to be virialized if it is dynamically relaxed (in equilibrium) so that the

virial theorem applies. However, as we go out from the center, at some point the galaxies are going to become sparse enough that the region is not in virial equilibrium. It is not possible to observationally determine the radius at which this happens. The virial radius is a rather arbitrary construct and is defined as the radius of a sphere centered on the center of the cluster, inside of which the density is 200 times the critical density of the Universe (which is $3H_0^2/8\pi G$, where H_0 is the Hubble constant).

Cosmology

Some general references for the cosmology section:

Dabrowski, M. P., & Stelmach, J. 1987, *Astrophysical Journal*, 94, 5, 1373.
Hogg, D. W. 2000, arXiv:astro-ph/9905116v4, http://arxiv.org/pdf/astro-ph/9905116v4.pdf
Weight, E. L. 2006, PASP, 118, 1711.
Christiansen, J. L., Siver, A. 2012, *American Journal of Physics*, 80, 5, 367.
P. J. E. Peebles, 1993, *Principles of Physical Cosmology*, (Princeton University Press), ISBN 978-0691019338.

Basic terminology and definitions

t_0: Cosmic time at the present epoch.
$\lambda_{\rm obs},\ \lambda_{\rm e}$: observed and emitted wavelength of light respectively.
z: Redshift. Measured directly from the observation of a shift in the wavelength of light from a distant object ($\lambda_{\rm obs}$), compared to what that wavelength would be here in a laboratory ($\lambda_{\rm e}$):

$$\lambda_{\rm obs} \ = \ (1+z)\lambda_{\rm e}$$

H_0: Hubble constant at the present epoch.
h_{70}: Hubble constant at the present epoch in units of 72 $\mathrm{km\ s^{-1}\ Mpc^{-1}}$.
$H(z)$, or just H: Hubble constant at an earlier epoch of the Universe, corresponding to a redshift of z.
$a(t)$: Scale factor for an expanding Universe (t is the age of the Universe; t_0 is the age now at the present epoch). By convention, $a(t_0) \equiv 1$, so in an expanding Universe, a was < 1 in the past. There is a simple relationship between $a(t)$ and z because *all* lengths change when the scale factor, changes, including *rulers*. The wavelength of light from distant objects is measured with today's rulers, so $a(t)\lambda_{\rm obs}$ = $\lambda_{\rm e}$, or

$$a(t) = \frac{1}{1+z}$$

t_H: $(= 1/H)$. This is the Hubble time, which is *not* the age of the Universe. $t_{H_0} = 1.40 \times 10^{10} h_{70}^{-1}$ yr, or $4.41 \times 10^{17} h_{70}^{-1}$ s.
l_H: $(= c/H)$, This is the Hubble length, $l_{H_0} \sim 4285.7 h_{70}^{-1}$ Mpc.
$\rho_c(t)$, $\rho_c(t_0)$: The critical density of the Universe when the age of the Universe is t, and t_0 (the current

epoch) respectively.

$$\rho_c(t) = \frac{3H^2}{8\pi G}$$
$$\rho_c(t_0) = \frac{3H_0^2}{8\pi G}$$
$$= 9.204 \times 10^{-30}\ h_{70}^2\ \mathrm{g\ cm}^{-3}$$
$$\sim 9.46 \times 10^{20}\ h_{70}^{-1}\ M_\odot\ l_H^{-3}$$

$\Omega_M = \rho_{\mathrm{M}}/\rho_c$: Baryonic and dark matter density ratio.
$\Omega_r = u_{\mathrm{r}}/(\rho_c c^2)$: Radiation energy density ratio.
Ω_Λ is the dark energy density:

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H^2}$$

Ω_k is the curvature parameter of the Universe. The current fashion is to assume that the total density parameter, Ω_{tot}, is exactly equal to 1 so that

$$\Omega_k = 1 - \Omega_M - \Omega_\Lambda - \Omega_r.$$

Friedmann equation The Friedmann equation results from using the Robertson-Walker metric (see below) in the solution of the GR field equations. One form of the Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\Omega_\Lambda + \Omega_k a^{-2} + \Omega_M a^{-3} + \Omega_r a^{-4}\right]$$

Remember that the current fashion is to whimsically set $\Omega_k = 1 - [\Omega_\Lambda + \Omega_M + \Omega_r]$. (Also note that we have not included other contributions to Ω, such as neutrinos.)

Instead of a, we write the R.H.S. in terms of z using $a^{-1} = (1+z)$. The R.H.S. of the above Friedmann equation is just $H^2(t)$ simply by definition of H. Then if we define

$$F(z) = \left[\Omega_\Lambda + \Omega_k(1+z)^2 + \Omega_M(1+z)^3 + \Omega_r(1+z)^4\right]^{\frac{1}{2}}$$

we simply have:

$$H(z) = H_0 F(z)$$

The function $F(z)$ comes up in the expressions for distance and time as described below.

Distance Measures

$$\text{Co-moving distance},\ D_c = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{F(z')}$$
$$\text{Proper distance},\ D_{\mathrm{proper}} = \frac{D_c}{(1+z)}$$
$$\text{Light travel-time distance}, D_{\mathrm{ltt}} = \frac{c}{H_0}\int_0^z \frac{\mathrm{d}z'}{(1+z')F(z')} \equiv c\, t_{\mathrm{look-back}}$$
$$\text{Angular diameter distance}, D_A = \frac{D_c}{1+z)}$$
$$\text{Luminosity distance}, D_L = (1+z)D_c\ =\ (1+z)^2 D_A$$

Note that the co-moving distance is also known as the conformal distance. See http://astrophysicsformulas.com for explanations of the meanings of all the above measures of distance. The look-back time, $t_{\rm look-back}$, is given below.

Time and age of the Universe

$$t_{\rm age\ now} = \frac{1}{H_0} \int_0^{\infty} \frac{{\rm d}z'}{(1+z')F(z')}$$

The age at some redshift z is

$$t_{\rm age\ at\ z} = \frac{1}{H_0} \int_z^{\infty} \frac{{\rm d}z'}{(1+z')F(z')}$$

The look-back time is the difference between the age of the Universe now and the age of the Universe at z:

$$t_{\rm look\ back} = \frac{1}{H_0} \int_0^{z} \frac{{\rm d}z'}{(1+z')F(z')}$$

Usage of the Luminosity Distance and the Redshift-Magnitude Relation
Define: $F =$ observed flux, $L =$ intrinsic luminosity at the source.

Bolometric flux and luminosity:

$$F \ = \ \frac{L}{4\pi D_L^2}$$

Monochromatic flux and luminosity:

$$F_\nu = \frac{L_{\nu(1+z)}}{L_\nu} \frac{L_\nu (1+z)}{4\pi D_L^2}$$

The first factor in the above is the spectrum-dependent 'K-correction.'

In νF_ν space:

$$\nu F_\nu = \frac{\nu_e L_{\nu_e}}{4\pi D_L^2}$$

In terms of magnitudes, using m and M for the apparent and absolute magnitudes respectively,

$$m = M + 5 \log \left(\frac{D_L}{10 \ {\rm pc}} \right) - 2.5 \log \left[\frac{(1+z) L_{\nu(1+z)}}{L_\nu} \right]$$

where the last term is the K-correction.

The last equation is the redshift-magnitude relation but this term also refers to the relation between observed flux and intrinsic luminosity for a given cosmology as a function of z (because the theoretical

D_L depends only on z and the cosmological model). If we know what the intrinsic luminosity of a source (at a redshift of z) should be, we can measure its observed flux and calculate D_L. However, for a given cosmology, D_L can also be predicted using the integral equation presented above, so the observed and predicted values can be compared for a sample of similar objects covering a range of z in order to try to distinguish between different cosmologies.

Robertson-Walker Metric
The Robertson-Walker metric is appropriate for an isotropic, pressureless medium that expands with a scale factor of $a(t)$.

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + R_0 \sin{(r/R_0)}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right] \qquad \text{positive curvature } (\kappa = +1)$$

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + r(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right] \qquad \text{no curvature, flat } (\kappa = 0)$$

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a^2(t)\left[\mathrm{d}r^2 + R_0 \sinh{(r/R_0)}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right] \qquad \text{negative curvature } (\kappa = -1)$$

where R_0 is the radius of curvature now (zero for the flat Universe case):

$$\begin{aligned} R_0 &= \frac{c}{H_0}\left(\Omega_{\mathrm{tot}} - 1\right)^{-\frac{1}{2}} \qquad \text{(positive curvature)} \\ &= 0 \quad \text{flat} \\ &= \frac{c}{H_0}\left(1 - \Omega_{\mathrm{tot}}\right)^{-\frac{1}{2}} \qquad \text{(negative curvature)} \end{aligned}$$

Cosmic Microwave Background (CMB) Temperature
Fixsen (2009, ApJ, 707, 916) used WMAP data to recalibrate old COBE data and obtained $T_{\mathrm{CMB}} = 2.72548 \pm 0.00057$ Kelvin. At earlier epochs,

$$T_{\mathrm{CMB}}(z) = (1+z)T_{\mathrm{CMB}}(z=0)$$

Concordance Cosmology Parameters

7-year WMAP main results: Jarosik et al. (2011), ApJ, 192, 14. This paper has a table summarizing the best measurements of all the key quantities, and also compares the WMAP-only values of these, as well as the corresponding values obtained when the WMAP data are combined with BAO and H_0 measurements. The WMAP-only results are separately described in Larson et al. (2011), ApJ, 192, 16. Komatsu et al. (2011) discuss some of the cosmological parameters as well (from WMAP+BAO+H) but give a lot of information on the temperature fluctuations.

Baryon acoustic oscillations (BAO): from galaxy distribution analysis; Percival et al. 2011 (MNRAS, 417, 3101), and Hubble constant measurements (Riess et al. 2009, ApJ, 699, 539).

Below is a selection of some of the key cosmological parameters from Jarosik et al. (2011) that assume a flat ΛCDM cosmology (i.e., $\Omega_{\rm tot} = 1$) and use WMAP data combined with information from Baryon Acoustic Oscillations from Sloan data and independent H_0 data . Consult the paper for parameters derived from deviations on the baseline model assumptions. Note that $h_{70} = [H_0 \text{ km s}^{-1} \text{ Mpc}^{-1}]/[70 \text{ km s}^{-1} \text{ Mpc}^{-1}]$. Likewise, $h \equiv [H_0 \text{ km s}^{-1} \text{ Mpc}^{-1}]/[100 \text{ km s}^{-1} \text{ Mpc}^{-1}]$. Further note that in the case of the baryon and dark matter densities, the directly measured quantities are $\Omega_b h^2$ and $\Omega_{DM} h^2$ respectively (i.e., with h^2, not without). The uncertainties are supposed to be 68% confidence. When directly comparing the values given below to the original source (Jarosik *et al.* 2011), don't forget to adjust the values for the factor $(h/h_{70})^2$ since the paper gives $\Omega_b h^2$ and $\Omega_{DM} h^2$, whereas we are using h_{70} instead of h.

	Parameter	Value
t_0	Age of the Universe	13.75 ± 0.11 Gyr
H_0	Hubble constant	$70.4^{+1.3}_{-1.4}\text{km s}^{-1} \text{ Mpc}^{-1}$
$\Omega_b h_{70}^2$	Baryon density measure	0.02234 ± 0.00052
Ω_b	Baryon density	0.02260 ± 0.00053
$\Omega_{DM} h_{70}^2$	Dark matter density measure	0.1110 ± 0.0035
Ω_{DM}	Dark matter density	0.1123 ± 0.0035
Ω_Λ	Dark energy density	$0.728^{+0.015}_{-0.016}$

10 Unit Conversions

The *Google* search tool is quite clever and can handle most unit conversions. However, some that are tricky (but commonly needed) are given below.

keV to Kelvin, Kelvin to keV

Temperature expressed in keV is not really a temperature, it is an energy. Use $kT_{\mathrm{Kelvin}}(\mathrm{Joules})/(1000e) =$ energy in keV, where k is Boltzmann's constant and e is the elementary charge. Thus:

$$\begin{aligned} T(\mathrm{Kelvin}) &= 1.16 \times 10^{7} \ \times \text{“Temperature” in keV} \\ \text{“Temperature” in keV} &= 8.625 \times 10^{-5}\, T\ (\mathrm{Kelvin}) \end{aligned}$$

Mass in kg, eV, MeV, and GeV

$$\begin{aligned} E(\mathrm{eV}) &= 5.6175 \times 10^{35}\, m(\mathrm{kg}) \\ E(\mathrm{MeV}) &= 5.6175 \times 10^{29}\, m(\mathrm{kg}) \\ E(\mathrm{GeV}) &= 5.6175 \times 10^{26}\, m(\mathrm{kg}) \\ m(\mathrm{kg}) &= 1.780 \times 10^{-36}\, E(\mathrm{eV}) \\ m(\mathrm{kg}) &= 1.780 \times 10^{-30}\, E(\mathrm{MeV}) \\ m(\mathrm{kg}) &= 1.780 \times 10^{-27}\, E(\mathrm{GeV}) \end{aligned}$$

Useful benchmarks:

Electron mass: ~ 511 keV
Proton mass: ~ 940 MeV or ~ 1 GeV.

keV to Angstroms, Angstroms to keV

$$\begin{aligned} E(\mathrm{keV}) &= \frac{12.3984190}{\lambda(\AA)} \\ \lambda(\AA) &= \frac{12.3984190}{E(\mathrm{keV})} \end{aligned}$$

Fundamental Constants and Solar System Data

astrophysicsformulas.com

The fundamental constants listed here are some of the most commonly used physical constants and are a subset of those recommended by the Committee on Data for Science and Technology (CODATA), and adopted by the National Institute of Science and Technology (NIST). For the full set, and excruciating details on methods, measurements, and uncertainties, consult the original paper at:

http://physics.nist.gov/cuu/Constants/Preprints/lsa2010.pdf (*CODATA Recommended Values of the Fundamental Physical Constants: 2010*, by Mohr, Taylor, and Newell, which was accepted for publication by *Reviews of Modern Physics*, in June 2012.)

If you just want to access the full (searchable) list of constants and data, go to:

http://physics.nist.gov/cuu/Constants/index.html

at NIST. Note that uncertainties are given by a number in parenthesis, and represent the uncertainties in the last digits of the quantity in question. Where no uncertainties are given for the fundamental constants (as opposed to astrophysical constants), the quantity is *defined* to have the exact value given. For example, the speed of light in contemporary physics is now *defined* to have the given value. The units in the table below are SI units except for "eV," and the atomic mass unit, both of which are accepted for use with SI units.

First, here are some reminders of common unit abbreviations.

A: Amperes
C: Coulombs
J: Joules
K: Kelvin
kg: kilograms
m: meters
N: Newtons
s: seconds
W: Watts (Joules per second)

Continue to next page →

Selection of Constants

Symbol	Description	Value	Units
e	elementary charge	$1.602176565(35) \times 10^{-19}$	C
eV	electron volt	$1.602176565(35) \times 10^{-19}$	J
h	Planck constant	$6.62606957(29) \times 10^{-34}$	J s
$\hbar$	Planck constant/2π	$1.054571726(47) \times 10^{-34}$	J s
c	speed of light in vacuum	2.99792458×10^{8}	$\mathrm{m\ s^{-1}}$
G	Newton's gravitation constant	$6.67384(80) \times 10^{-11}$	$\mathrm{m^3\ kg^{-1}\ s^{-2}}$
ϵ_0	permittivity of free space	$8.854187817 \times 10^{-12}$	$\mathrm{F\ m^{-1}}$
μ_0	permeability of free space	$12.566370614 \times 10^{-7}$	$\mathrm{N\ A^{-2}}$
α	fine structure constant	$7.2973525698(24) \times 10^{-3}$	dimensionless
α^{-1}	inverse fine structure constant	$137.035999074(44)$	dimensionless
R_∞	Rydberg constant	$10973731.568539(55)$	$\mathrm{m^{-1}}$
m_p	proton mass	$1.672621777(74) \times 10^{-27}$	kg
m_e	electron mass	$9.10938291(40) \times 10^{-31}$	kg
m_p/m_e	proton to electron mass ratio	$1836.15267245(75)$	dimensionless
u= $\mathrm{m}[^{12}\mathrm{C}]/12$	unified atomic mass unit	$1.660538921(73) \times 10^{-27}$	kg
k	Boltzmann constant	$1.3806488(13) \times 10^{-23}$	J K^{-1}
N_A	Avogadro constant	$6.02214129(27) \times 10^{23}$	$\mathrm{mol^{-1}}$
$R = N_A k$	molar gas constant	$8.3144621(75)$	J $\mathrm{mol^{-1}\ K^{-1}}$
σ	Stefan-Boltzmann constant	$5.670373(21) \times 10^{-8}$	W $\mathrm{m^{-2}\ K^{-4}}$

Continue to next page $\rightarrow$

Data from Solar System Measurements

Note: for solar system objects that are not mentioned on this page, you can consult solarsystem.nasa.gov for quick answers but beware that the NASA website does not give sources or uncertainties for the data. If your application involves serious research (as opposed to, for example, making or answering exam questions), then you will need to investigate the origins and uncertainties of the data by yourself in the scientific literature.

A 2009 IAU report at http://www.springerlink.com/content/t855371133q54g77/ gives, in painful detail, adopted values of various astronomical quantities. Less painful summaries can be found at http://maia.usno.navy.mil/NSFA/IAU2009_consts.html#d.

Astronomical Unit (AU)

The value recommended in the 2009 IAU report is given as

$$1\ \mathrm{AU} = 1.49597870700 \pm 0.00000000003 \times 10^{11}\ \mathrm{m}$$

(i.e., the uncertainty is 3 m). The reference cited for this is Pitjeva and Standish (2009: DOI 10.1007/s10569-009-9203-8) at http://www.springerlink.com/content/21885q7262104u76/fulltext.pdf

Solar Luminosity

Noerdlinger (2008, at http://adsabs.harvard.edu/abs/2008arXiv0801.3807N) quotes Bahcall's 1989 book, *Neutrino Astrophysics* (ISBN: 978-0521379755), with

$$L_\odot = 3.939 \times 10^{33}\ \mathrm{erg\ s^{-1}}$$

for the solar luminosity, which includes a 2.3% contribution from neutrinos, and

$$L_\odot = 3.856 \times 10^{33}\ \mathrm{erg\ s^{-1}}$$

without the neutrino contribution. According to Sofia and Li (2000, at http://arxiv.org/abs/astro-ph/0010428), the solar luminosity had not varied by more than 0.1% in the last 2 to 3 centuries.

The value given for the solar luminosity at http://solarsystem.nasa.gov/planets/ is $3.83 \times 10^{33}\ \mathrm{erg\ s^{-1}}$, which is closest to the value without the neutrino contribution. No sources are given by NASA for the value on its website.

Solar Radius

The value obtained from Mercury transits by Emilio et al. (2012, at http://arxiv.org/abs/1203.4898) is

$R_\odot = 6.96342 \pm 0.00065 \times 10^8$ meters.

The value given at http://solarsystem.nasa.gov/planets/ is:

$R_\odot = 6.95508 \times 10^8$ meters

and no error or uncertainty is given.

Solar Mass

The solar mass is obtained from applying Newton's and Kepler's laws for orbital motion of the Earth, using the measured orbital period and the semimajor axis of the orbit, along with an independently measured value of Newton's gravitational constant. A reference paper for the solar mass value by Castellini et al. (1996) at http://arxiv.org/pdf/astro-ph/9606180v2.pdf itself quotes the 1994 Astronomical Almanac, with a value of

$M_{\odot} = 1.98892 \pm 0.00025 \times 10^{30}$ kg.

The value given at http://solarsystem.nasa.gov/planets/ is 1.9891×10^{30} kg. The NASA website does not give sources for most of the numbers appearing on its website.

Other Solar Quantities

Other key quantities given at http://solarsystem.nasa.gov/planets/ are:

Spectral type: G2V
Density: 1.409 g cm^{-3}
Surface gravity: 274.0 m s^{-2}
Effective surface temperature: 5777 Kelvin

Solar System Objects: Radii, Masses, and Densities

The values below in the table for the sizes, masses, and densities of Solar System planets and Pluto are from http://solarsystem.nasa.gov. The NASA website provides no sources or uncertainties for the data: if your application is serious research please investigate the current scientific literature by yourself. See also earlier notes on the radius and mass of the Sun.

Continue to next page $\rightarrow$

Table 1: Radii of Solar System Objects

Name	R (Radius in km)	(R/R_E) (Ratio to Earth Radius)
Earth	6731.0	1.00
Moon	1737.5	0.2727
Mercury	2439.7	0.3829
Venus	6061.8	0.9499
Mars	3389.5	0.04848
Jupiter	69911.0	10.9733
Saturn	58232.0	9.1402
Uranus	25362.0	3.9809
Neptune	24622.0	3.8647
Pluto	1151.0	0.1807

Table 2: Masses of Solar System Objects

Name	M (Mass in kg)	(M/M_E) (Ratio to Earth Mass)
Earth	5.9722×10^{24}	1.00
Moon	7.3477×10^{22}	0.0123
Mercury	3.3010×10^{23}	0.055
Venus	4.8673×10^{24}	0.815
Mars	6.4169×10^{23}	0.107
Jupiter	1.8981×10^{27}	317.828
Saturn	5.6832×10^{26}	95.161
Uranus	8.6810×10^{25}	14.536
Neptune	1.0241×10^{26}	17.148
Pluto	1.309×10^{22}	0.002192

Table 3: Densities of Solar System Objects

Name	ρ (Density in $\mathrm{g\ cm^{-3}}$)	(ρ/ρ_E) (Ratio to Earth Density)
Earth	5.513	1.00
Moon	3.344	0.6066
Mercury	5.427	0.9844
Venus	5.243	0.9510
Mars	3.934	0.7136
Jupiter	1.326	0.2405
Saturn	0.687	0.1246
Uranus	1.270	0.2304
Neptune	1.638	0.2971
Pluto	2.050	0.3718