The Ninth Grade Math Competition Class

Logarithm Challenging Problems

Anthony Wang

0. What is the logarithm of $27\sqrt[4]{9}\sqrt[3]{9}$ base 3?

$$\log_3(27\sqrt[4]{9}\sqrt[3]{9}) = \log_3(3^3\sqrt[4]{3^2}\sqrt[3]{3^2})$$

$$= \log_3(3^3\, 3^{\frac{2}{4}}\, 3^{\frac{2}{3}})$$

$$\log_3(3^x) = x$$

$$= \log_3(3^{3+\frac{1}{2}+\frac{2}{3}})$$

$$= \log_3(3^{\frac{25}{6}}) = \boxed{\frac{25}{6}}$$

1. Find x if $\log_9(2x-7) = \frac{3}{2}$.

$$\log_a b = c \iff a^c = b$$

$$\underbrace{(3^2)^{\frac{3}{2}} = 9^{\frac{3}{2}}}_{=3^3} = 2x-7$$

$$3^3 = 2x-7$$
$$27 = 2x-7$$
$$2x = 34 \qquad \boxed{x=17}$$

2. Find $\log_{\sqrt{3}} \sqrt[3]{9}$.

3^2

$$\log_{3^a}(3^b) = \frac{b}{a} \iff (3^a)^{\frac{b}{a}} = 3^b$$

$$\log_{3^{\frac{1}{2}}}(3^{\frac{2}{3}}) = \frac{\frac{2}{3}}{\frac{1}{2}} = \boxed{\frac{4}{3}}$$

3. Solve the equation $\log_{2x} \overset{6^3}{216} = x$, where x is real.

$$\boxed{x=3}$$

$$(\underset{\Uparrow}{2x})^{\overset{3}{\downarrow}x} = 216$$

$$2 \cdot 3 = 6$$

4. Find base b such that $\log_b 5\sqrt{5} = \frac{5}{2}$.

$$\left(b^{\frac{5}{2}}\right)^{\frac{2}{5}} = \left(5\sqrt{5}\right)^{\frac{2}{5}}$$

$$b = \left(5^{\frac{3}{2}}\right)^{\frac{2}{5}} = \boxed{5^{\frac{3}{5}} = \sqrt[5]{125}}$$

5. If $\log_2 b - \log_2 a = 3$, then $b^2 - a^2 = Ma^2$, compute M.

$$\log_2\left(\frac{b}{a}\right) = 3 \qquad 8 = \frac{b}{a}$$

$$64 = 8^2 = \left(\frac{b}{a}\right)^2 = \frac{b^2}{a^2}$$

$$\frac{b^2}{a^2} - 1 = \boxed{M = 63}$$

$$\uparrow$$

$$64$$

6. If $\frac{\log_b a}{\log_c a} = \frac{19}{99}$, $\frac{b}{c} = c^k$, find the value of k.

$$\boxed{\frac{\log_a b}{\log_a c} = \log_c b}$$

$$\log_2 3 = \frac{\log_{10} 3}{\log_{10} 2}$$

$$\log_a b = \frac{1}{\log_b a}$$

$$b^{\frac{19}{99}} = c$$

$$\frac{\log_b a}{\log_c a} = \frac{\log_a c}{\log_a b} = \log_b c = \frac{19}{99}$$

$$b = c^{\frac{99}{19}}$$

$$\Rightarrow \frac{b}{c} = c^{\frac{80}{19}}$$

$$\boxed{k = \frac{80}{19}}$$

7. Let $T = 1.8$, compute base b if $\log_b(75T) = 2 - \log_b 3 + \log_b 5$.

$\log_b(b^2) = 2$

actually plus!

$$\log_b(75T) = \log_b(b^2) + \log_b 3 + \log_b 5$$

$$\cancel{\log_b}(75T) = \cancel{\log_b}(15b^2)$$

$$75T = 15b^2$$

$$5T = b^2$$

$$b = \sqrt{5T} \quad T = 1.8$$

$$\boxed{b = 3}$$

8. If $\log_{225} x + \log_x 15 = \frac{11}{6}$, find x.

$$\frac{1}{\log_x 225} + \log_x 15 = \frac{11}{6}$$

$$\frac{1}{\log_x(15^2)} + \log_x 15 = \frac{11}{6}$$

$$\frac{1}{2\log_x 15} + \log_x 15 = \frac{11}{6}$$

$$y = \log_x 15$$

$$\frac{1}{2y} + y = \frac{11}{6}$$

$$\frac{1}{2} + y^2 = \frac{11}{6} y$$

$$3 + 6y^2 - 11y = 0$$

$$(3y - 1)(2y - 3) = 0$$

$$y = \frac{1}{3}, \frac{3}{2}$$

$$\frac{1}{3} = \log_x 15 \Rightarrow x^{\frac{1}{3}} = 15 \Rightarrow \boxed{x = 15^3}$$

$$\frac{3}{2} = \log_x 15 \Rightarrow x^{\frac{3}{2}} = 15 \Rightarrow \boxed{x = 15^{\frac{2}{3}}}$$

9. Evaluate $\frac{1}{\log_2 \frac{1}{6}} - \frac{1}{\log_3 \frac{1}{6}} - \frac{1}{\log_4 \frac{1}{6}}$

$$\log_{\frac{1}{6}} 2 - \log_{\frac{1}{6}} 3 - \log_{\frac{1}{6}} 4 = \log_{\frac{1}{6}} \frac{2}{3 \cdot 4} = \boxed{\log_{\frac{1}{6}} \frac{1}{6} = 1}$$

10. Compute the value of N for which $\frac{1}{\log_2 100} + \frac{1}{\log_3 100} + \frac{1}{\log_6 100} + \frac{1}{\log_9 100} = \frac{2}{\log_N 100}$.

$$\frac{1}{\log_2 100} = \log_{100} 2 \qquad \log_{100} 2 + \log_{100} 3 + \log_{100} 6 + \log_{100} 9$$

$$= 2 \log_{100} N$$

$$\log_{100} a + \log_{100} b = \log_{100} ab$$

Apply Rule

$$\log_{100}(2 \cdot 3 \cdot 6 \cdot 9) = 2 \log_{100} N$$

$$\cancel{\log_{100}}(2 \cdot 3 \cdot 6 \cdot 9) = \cancel{\log_{100}}(N^2)$$

$$2 \log_{100} N = \log_{100}(N^2)$$

$$2 \log_{100} N = \log_{100} N + \log_{100} N = \log_{100}(N \cdot N)$$

$$\Rightarrow 2 \cdot 3 \cdot 6 \cdot 9 = N^2$$

$$\boxed{N = 18}$$

11. Given the points $A(\log 2, \log 3)$ and $B(\log(\log T^2), \log(\log T^3))$, compute the slope of the line $\overleftrightarrow{AB}$.

$(a, b) \quad (c, d)$

$$\text{Slope} = \frac{b-d}{a-c} = \frac{\log 3 - \log(\log(T^3))}{\log 2 - \log(\log(T^2))}$$

$$= \frac{\log\left(\frac{3}{\log(T^3)}\right)}{\log\left(\frac{2}{\log(T^2)}\right)} = \frac{\log\left(\frac{\cancel{3}}{\cancel{3}\log T}\right)}{\log\left(\frac{\cancel{2}}{\cancel{2}\log T}\right)} = 1$$

12. Given that $\log_6 a + \log_6 b + \log_6 c = 6$, and a, b, c are positive integers that form an increasing geometric sequence and $b - a$ is the square of an integer. Find $a + b + c$.

$\log_6 abc = 6$

$$\boxed{abc = 6^6}$$

Sequence

1, 3, 6, 10,

Geometric Seq.

$1 \xrightarrow{\times 3} 3 \xrightarrow{\times 3} 9 \xrightarrow{\times 3} 27$

$\frac{3}{2} \xrightarrow{\times \frac{2}{3}} 1 \xrightarrow{\times \frac{2}{3}} \frac{2}{3} \xrightarrow{\times \frac{2}{3}} \frac{4}{9}$

$a \xrightarrow{\times R} b \xrightarrow{\times R} c$

$b - a = x^2$

$Ra = 6^2 = b = 36$

a, b, c

$a, Ra, R^2 a$

$R^3 a^3 = 6^6$

$36 - a = x^2$

$36 - x^2 = a$

$1^2 = 1$ ~~$a = 35$~~

$2^2 = 4$ ~~$a = 32$~~

$3^2 = 9$ $a = 27$

$4^2 = 16$

$5^2 = 25$

$3^3 \cdot 2^2 \cdot 3^2 \cdot c = 2^6 \cdot 3^6$

$\Rightarrow a = 27, b = 36,$

$c = 2^4 \cdot 3 = 48$

$$\boxed{a + b + c = 27 + 36 + 48 = 111}$$