The Ninth Grade Math Competition Class
Factorization
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1. Two non-zero real numbers, a and b, satisfy $ab = a - b$. Find all possible values of $\frac{a}{b} + \frac{b}{a} - ab$.

$$\frac{a}{b} + \frac{b}{a} - ab = \frac{a^2+b^2-a^2b^2}{ab}$$

$$= \frac{a^2+b^2-(a-b)^2}{ab}$$

$$= \frac{\cancel{a^2}+\cancel{b^2}-\cancel{a^2}+2\cancel{ab}-\cancel{b^2}}{\cancel{ab}} = 2$$

2. Without a calculator, find the sum of the digits of the number $2003^4 - 1997^4$.

$$2003^4 - 1997^4 = (2003^2 - 1997^2)(2003^2 + 1997^2)$$

$$(2000+3)^2 + (2000-3)^2$$

$$2000^2 + \cancel{2\cdot 2000\cdot 3} + 3^2 + 2000^2 - \cancel{2\cdot 2000\cdot 3} + 3^2$$

$$= (2003-1997)(2003+1997)(8000{,}018)$$

$$6 \cdot 4000 \cdot 8000{,}018$$

$$(2000+3)^4 - (2000-3)^4$$

$$\cancel{2000^4} + \binom{4}{1}2000^3\cdot 3 + \cancel{\binom{4}{2}2000^2\cdot 3^2} + \binom{4}{3}2000\cdot 3^3 + \cancel{3^4}$$

$$\cancel{-2000^4} + \binom{4}{1}2000^3\cdot 3 - \binom{4}{2}2000^2\cdot 3^2 + \binom{4}{3}2000\cdot 3^3 - 3^4$$

$$8\cdot 2000^3\cdot 3 + 4\cdot 2000\cdot 3^3$$

3. Express $2^{22}+1$ as the product of two four-digit numbers.

$$4a^4+b^4= \qquad b=1$$

$$a=2^{10}$$

$$\left(2^{22}+1+2\cdot 2^{11}\cdot 1\right)-2\cdot 2^{11}\cdot 1$$

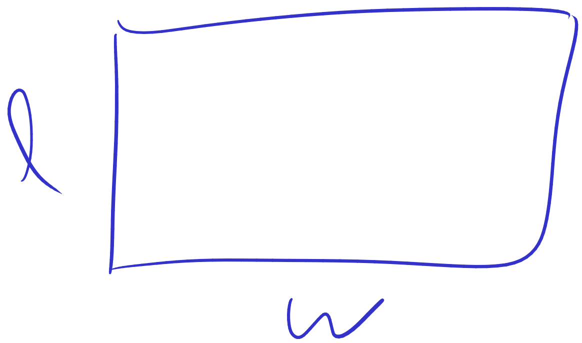
$$a^2+b^2$$

$$(a+b)^2 \qquad 2ab$$

$$\left(2^{11}+1\right)^2-\underbrace{2\cdot 2^{11}\cdot 1}_{2^{12}}$$

$$=\left(2^{11}+1-2^6\right)\left(2^{11}+1+2^6\right)$$

4. Find the length and the width of a rectangle with integer sides whose area is equal to its perimeter.



$$lw = 2l + 2w$$

$$lw - 2l - 2w = 0$$

$$lw - 2l - 2w + 4 = 4$$

$$(l-2)(w-2) = 4$$

1	4	$\Rightarrow l = 3,\ w = 6$
2	2	$\Rightarrow l = 4,\ w = 4$
4	1	$\Rightarrow l = 6,\ w = 3$

5. Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?

$\cancel{21}$, $\cancel{60}$, 119, $\cancel{180}$, $\cancel{231}$

5, 7, 11, 13, 17

$(2a+1)(2b+1) - 2a - 1 - 2b - 1$

$4ab + \cancel{2a} + \cancel{2b} + 1 - \cancel{2a} - 1 - \cancel{2b} - 1$

$4ab - 1$

$13 \cdot 17 - 13 - 17 = 221 - 13 - 17 = 191$

$11 \cdot 13 - 11 - 13 = 143 - 24 = 119$

(A) 21 (B) 60 (C) 119 (D) 180 (E) 231

6. m, n are integers such that $m^2 + 3m^2n^2 = 30n^2 + 517$. Find $3m^2n^2$.

$$3m^2n^2 + m^2 - 30n^2 \overset{-10}{=} 517^{-10}$$

$$(m^2 - 10)(3n^2 + 1) = 507 = 3 \cdot 13^2$$

$3 \cdot 13$ 13

$m^2 - 10 = 3 \cdot 13$ $m^2 = 49$ $m = 7$

$3n^2 + 1 = 13$ $3n^2 = 12$ $n = 2$

588

7. How many distinct ordered pairs of positive integers (m, n) are there so that the sum of the reciprocals of m and n is $\frac{1}{4}$?

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{4}$$

$$\frac{m+n}{mn} = \frac{1}{4}$$

$$4m + 4n = mn$$

$$0 = mn - 4m - 4n$$

$$16 = mn - 4m - 4n + 16$$

$$16 = (m-4)(n-4)$$

$$\left.\begin{array}{cc} 1 & 16 \\ 2 & 8 \\ 4 & 4 \\ 8 & 2 \\ 16 & 1 \end{array}\right\} \text{5 Solutions}$$

8. Find all prime factors of $3^{18} - 2^{18}$

$$(3^9 - 2^9)(3^9 + 2^9)$$

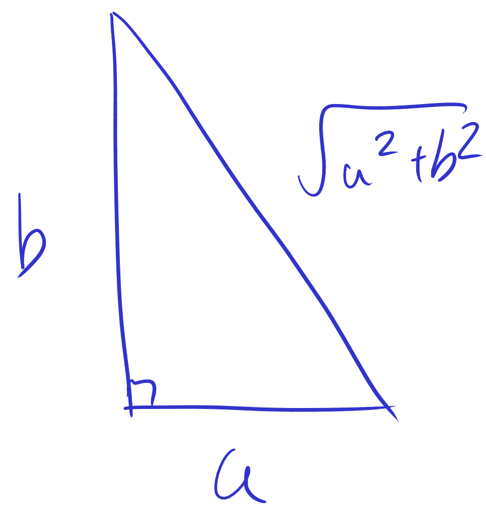
$$(3^3 - 2^3)(3^6 + 3^3 \cdot 2^3 + 2^6)(3^3 + 2^3)(3^6 - 3^3 2^3 + 2^6)$$

$$19 \cdot (729 + 27 \cdot 8 + 64) \cdot 35 \cdot (729 - 27 \cdot 8 + 64)$$

$$\boxed{19 \cdot 1009 \cdot 5 \cdot 7 \cdot 577}$$

9. An $m \times n \times p$ rectangular box has half the volume of an $(m + 2) \times (n + 2) \times (p + 2)$ rectangular box, where $m, n,$ and p are integers, and $m \leq n \leq p$. What is the largest possible value of p?

10. How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?



$$\frac{ab}{2} = 3\left(a+b+\sqrt{a^2+b^2}\right)$$

$$ab = 6a+6b+6\sqrt{a^2+b^2}$$

$$ab-6a-6b = 6\sqrt{a^2+b^2}$$

$$(ab-6a-6b)^2 = 36a^2+36b^2$$

$$a^2b^2-12a^2b-12ab^2+\cancel{36a^2}+\cancel{36b^2}+72ab$$

$$=\cancel{36a^2}+\cancel{36b^2}$$

$$ab-12a-12b+72=0$$

$$ab-12a-12b+144=72$$

$(a-12)(b-12)=72$

$2^3 \cdot 3^2$

$4 \cdot 3 = 12$

1, 72

13, 84

72, 1

84, 13

$\frac{12}{2} = 6$

11. The integer N is positive. There are exactly 2005 pairs (x, y) of positive integers satisfying:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}$$

Prove that N is a perfect square.

$$\frac{x+y}{xy} = \frac{1}{N}$$

$$Nx + Ny = xy$$

$$0 = xy - Nx - Ny$$

$$N^2 = xy - Nx - Ny + N^2$$

$$N^2 = (x-N)(y-N)$$

$$2005 = 5 \cdot 401$$

$$N^2 = a^{2004} \qquad N = a^{1002}$$

$$\text{or } N^2 = a^4 \cdot b^{400} \qquad N = a^2 \cdot b^{200}$$