The Ninth Grade Math Competition Class

Logarithm Challenging Problems

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0. What is the logarithm of $27\sqrt[4]{9}\sqrt[3]{9}$ base 3?

$$\log_3\left(27\sqrt[4]{9}\sqrt[3]{9}\right)$$

$$\log_3\left(3^3\left(3^2\right)^{\frac{1}{4}}\left(3^2\right)^{\frac{1}{3}}\right) = \log_3\left(3^3\, 3^{\frac{1}{2}}\, 3^{\frac{2}{3}}\right)$$

$$= \log_3\left(3^{\frac{25}{6}}\right) = \boxed{\frac{25}{6}}$$

1. Find x if $\log_9(2x - 7) = \frac{3}{2}$.

$$9^{\frac{3}{2}} = 2x - 7$$

$$(3^2)^{\frac{3}{2}} = 2x - 7$$

$$27 = 2x - 7$$

$$\Rightarrow \boxed{x = 17}$$

2. Find $\log_{\sqrt{3}} \sqrt[3]{9}$.

$$\log_{3^{\frac{1}{2}}} 3^{\frac{2}{3}} = \frac{\frac{2}{3}}{\frac{1}{2}} = \boxed{\frac{4}{3}}$$

3. Solve the equation $\log_{2x} 216 = x$, where x is real.

$(2x)^x = 216 = 6^3$

$\parallel$

$2^3 3^3$

$\boxed{x = 3}$

4. Find base b such that $\log_b 5\sqrt{5} = \frac{5}{2}$.

$$b^{\frac{5}{2}} = 5\sqrt{5}$$

$$b^{\frac{5}{2}} = 5^1 5^{\frac{1}{2}}$$

$$b^{\frac{5}{2}} = 5^{\frac{3}{2}}$$

$$b = \left(5^{\frac{3}{2}}\right)^{\frac{2}{5}} = \boxed{5^{\frac{3}{5}}}$$

5. If $\log_2 b - \log_2 a = 3$, then $b^2 - a^2 = Ma^2$, compute M.

$\log_2 \frac{b}{a} = 3$

$2^3 = \frac{b}{a}$

$8a = b$

$(8a)^2 - a^2 = Ma^2$

$64a^2 - a^2 = Ma^2$

$63\cancel{a^2} = M\cancel{a^2}$

6. If $\frac{\log_b a}{\log_c a} = \frac{19}{99}$, $\frac{b}{c} = c^k$, find the value of k.

$$\frac{\log_a b}{\log_a c} = \log_c b$$

$$\log_b a = \frac{1}{\log_a b}$$

$$\log_c a = \frac{1}{\log_a c}$$

$$\frac{\frac{1}{\log_a b}}{\frac{1}{\log_a c}} = \frac{19}{99}$$

$$\frac{\log_a c}{\log_a b} = \frac{19}{99}$$

$$\log_b c = \frac{19}{99}$$

$$\frac{b}{c} = c^k$$

$$\frac{c^{\frac{99}{19}}}{c^1} = c^k$$

$$c^{\frac{80}{19}} = c^k$$

$$\left(b^{\frac{19}{99}}\right)^{\frac{99}{19}} = c^{\frac{99}{19}}$$

$$b = c^{\frac{99}{19}}$$

$$\boxed{b^{\frac{19}{99}} = c}$$

7. Let $T = 1.8$, compute base b if $\log_b(75T) = 2 + \log_b 3 + \log_b 5$.

$$\log_b(135) = 2 + \log_b 3 + \log_b 5$$

$$= 2 + \log_b 3\cdot 5$$

$$= \log_b b^2 + \log_b 3\cdot 5$$

$$\log_b 135 = \log_b b^2 \cdot 15$$

$$135 = 15b^2 \Rightarrow b = 3$$

8. If $\log_{225} x + \log_x 15 = \frac{11}{6}$, find x.

$= 15^2$

$$\log_{(15^2)} x + \log_x 15 = \frac{11}{6}$$

$$\frac{1}{2}\log_{15} x + \log_x 15 = \frac{11}{6}$$

$$\frac{1}{3} = \log_x 15 \Rightarrow x^{\frac{1}{3}} = 15$$

$$x = 15^3$$

$$\log_{15} x = \frac{1}{\log_x 15}$$

$$y = \log_x 15$$

$$\frac{3}{2} = \log_x 15$$

$$x^{\frac{3}{2}} = 15$$

$$x = 15^{\frac{2}{3}}$$

$$\frac{1}{2\log_x 15} + \log_x 15 = \frac{11}{6}$$

$$(3y-1)(2y-3) = 0$$

$$\frac{1}{2y} + y = \frac{11}{6}$$

$$y = \frac{1}{3} \quad y = \frac{3}{2}$$

$$\frac{1}{2} + y^2 = \frac{11}{6}y$$

$$6y^2 - 11y + 3 = 0$$

$\frac{1}{\log_a b} = \log_b a$; $\log_{\frac{1}{6}} 2 + \log_{\frac{1}{6}} 3 + \log_{\frac{1}{6}} 4 = \log_{\frac{1}{6}} \frac{2}{3 \cdot 4} = \log_{\frac{1}{6}} \frac{1}{6}$

9. Evaluate $\frac{1}{\log_2 \frac{1}{6}} - \frac{1}{\log_3 \frac{1}{6}} - \frac{1}{\log_4 \frac{1}{6}}$

$= 1$

10. Compute the value of N for which $\frac{1}{\log_2 100} + \frac{1}{\log_3 100} + \frac{1}{\log_6 100} + \frac{1}{\log_9 100} = \frac{2}{\log_N 100}$.

$\log_{100} 2 + \log_{100} 3 + \log_{100} 6 + \log_{100} 9$

$= 2 \log_{100} N$

$\log_{a^m}(b^n) = \frac{n}{m} \log_a b$

$\log_{100} 2 \cdot 3 \cdot 6 \cdot 9 = 2 \log_{100} N$

$\log_{100} 2 \cdot 3 \cdot 6 \cdot 9 = \log_{100} N^2$

$2 \cdot 3 \cdot 6 \cdot 9 = N^2$

$N = 18$

11. Given the points $A\,(\log 2, \log 3)$ and $B\,(\log(\log T^2), \log(\log T^3))$, compute the slope of the line $\overleftrightarrow{AB}$.

$$\frac{\log 3 - \log(\log T^3)}{\log 2 - \log(\log T^2)}$$

$$\frac{\log \frac{3}{\log T^3}}{\log \frac{2}{\log T^2}} = \frac{\log \frac{\cancel{3}}{\cancel{3}\log T}}{\log \frac{\cancel{2}}{\cancel{2}\log T}} = \frac{\cancel{\log \frac{1}{\log T}}}{\cancel{\log \frac{1}{\log T}}}$$

$$= 1$$

12. Given that $\log_6 a + \log_6 b + \log_6 c = 6$, and a, b, c are positive integers that form an increasing geometric sequence and $b - a$ is the square of an integer. Find $a + b + c$.

$2 \xrightarrow{3} 6 \xrightarrow{3} 18 \xrightarrow{\times 3} 54 \xrightarrow{3} 162$

$a \xrightarrow{r} ar \xrightarrow{r} ar^2 \xrightarrow{r} ar^3$

$\log_6 a + \log_6 ar + \log_6 ar^2 = 6$

$\log_6 a \cdot ar \cdot ar^2 = 6$

$a^3 r^3 = 6^6$

$ar = 6^2 = 36$

$ar - a = x^2$

$36 - a = x^2$ 1, 4, 9, 16, 25

$x = 1$

$\Rightarrow a = 35$

$x = 9$

$a = 27$

$r = \frac{36}{27} = \frac{4}{3}$

$ar^2 = 48$

$a = 27$, $b = 36$, $c = 48$

$27 + 36 + 48 = 111$