The Ninth Grade Math Competition Class
Complex Numbers
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z $\bar{z}$ -8

1. Suppose $(-3+8i)(-3+Ai)$ is a real number, find the value of A where A is real.

$(-3+8i)(-3-8i) = 9 + 64 = 73$

2. Find all complex numbers whose squares equal $7 - 24i$.

$$z^2 = 7 - 24i$$

$$z = a + bi$$

$$(a+bi)^2 = 7 - 24i$$

$$a^2 + 2abi + b^2 \underbrace{i^2}_{-1} = 7 - 24i$$

$$a^2 - b^2 + 2abi = 7 - 24i$$

$$\overset{4}{a}^2 - \overset{-3}{b}^2 = 7$$

$$2ab = -24$$

$4, -3$

$-4, 3$

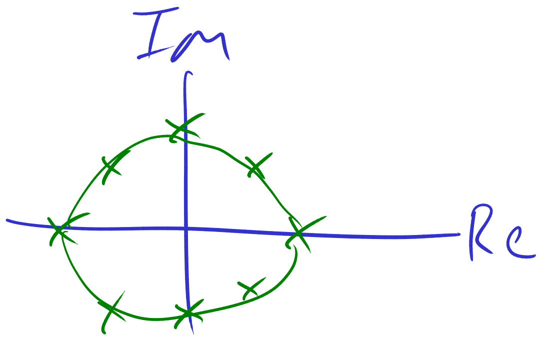
$$\boxed{\begin{matrix} 4 - 3i \\ -4 + 3i \end{matrix}}$$

3. Let $a = \frac{(2+i)^2}{3+i}$, find $1 + \frac{1}{a}$.

$$1 + \frac{3+i}{(2+i)^2} = 1 + \frac{3+i}{4+2i-1} = 1 + \frac{3+i}{3+2i} = \frac{3+2i+3+i}{3+2i}$$

$$= \frac{6+3i}{3+2i} \frac{(3-2i)}{(3-2i)} = \frac{18+9i-12i+6}{9+4} = \boxed{\frac{24-3i}{13}}$$

4. Find all x such that $x^5 = x^3$ (What if $x^5 = x^{-3}$).



$$x^5 = x^3$$

$$x^5 - x^3 = 0$$

$$x^3(x^2-1) = 0$$

$$\underbrace{x^3}_{0} \text{ or } \underbrace{(x^2-1)}_{0}$$

$$x^2 - 1 = 0 \quad x^2 = 1 \quad \boxed{x = \pm 1}$$

$$x^5 = \underbrace{x^{-3}}_{\frac{1}{x^3}}$$

$$x^5 = \frac{1}{x^3}$$

$$x^8 = 1$$

$$x^8 - 1 = 0$$

$$(x^4-1)(x^4+1) = 0$$

$$(x^2-1)(x^2+1)(x^4+1) = 0$$

$$(x-1)(x+1)(x^2+1)(x^4+1) = 0$$

$$x^2+1=0$$

$$x^2 = -1 \quad x = \pm i$$

$$x^4+1=0$$

$$(x^2)^2 - (i)^2 = 0$$

$$(x^2-i)(x^2+i) = 0$$

$$x^2 = i$$

$$x^2 = -i$$

$$x^2 = i$$

$$x = a + bi$$

$$(a+bi)^2 = i$$

$$a^2 + 2abi - b^2 = i$$

$$a^2 = b^2$$

$$a^2 - b^2 = 0$$

$$2ab = 1$$

$$a = b \quad 2a^2 = 1 \quad a = \frac{\pm\sqrt{2}}{2}$$

$$a = -b \quad -2a^2 = 1 \quad a = \frac{\pm\sqrt{2}}{2}$$

$$x = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$$

$$x = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$$

5. If $x = \frac{1-\sqrt{3}i}{2}$, what is $\frac{1}{x^2-x}$.

$$\frac{1}{x(x-1)} = \frac{1}{\left(\frac{1-\sqrt{3}i}{2}\right)\left(\frac{-1-\sqrt{3}i}{2}\right)} = \frac{1}{-\frac{1}{4}-\frac{3}{4}} = \frac{1}{-1} = -1$$

$$(a+b)(-a+b) = -a^2+b^2$$

$$a = \frac{1}{2}$$

$$b = -\frac{\sqrt{3}}{2}i$$

6. Show that $\overline{w+z} = \bar{w} + \bar{z}$, and $\overline{wz} = \bar{w} \cdot \bar{z}$.

$w = a + bi = 2 + 3i$

$\bar{w} = \overline{a+bi} = a - bi = 2 - 3i$

$w\bar{w} = (a+bi)(a-bi)$
$= a^2 + b^2$

$w = a + bi$
$z = c + di$

$\overline{a+bi+c+di} = \overline{a+bi} + \overline{c+di}$

$a + c - (b+d)i = a - bi + c - di$ ✓

$\overline{(a+bi)(c+di)} = (\overline{a+bi})(\overline{c+di})$

$\overline{ac + bci + adi - bd} = (a-bi)(c-di)$

$ac - bd - (bc + ad)i = ac - bd - adi - bci$ ✓

7. Write $\sqrt{-16+30i}$ as a complex number.

$$\sqrt{-16+30i} = a+bi$$

$$-16+30i = a^2-b^2+2abi$$

$$-16 = a^2 - b^2$$

$$30 = 2ab$$

3, 5

−3, −5

$3+5i$

$-3-5i$

$= -(3+5i)$

8. A function f is defined on the complex numbers by $f(z) = (a + bi)z$, where a and b are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a + bi| = 8$ and that $b^2 = \frac{m}{n}$, where m and n are positive integers, Find $m + n$.

$f(z)$ z

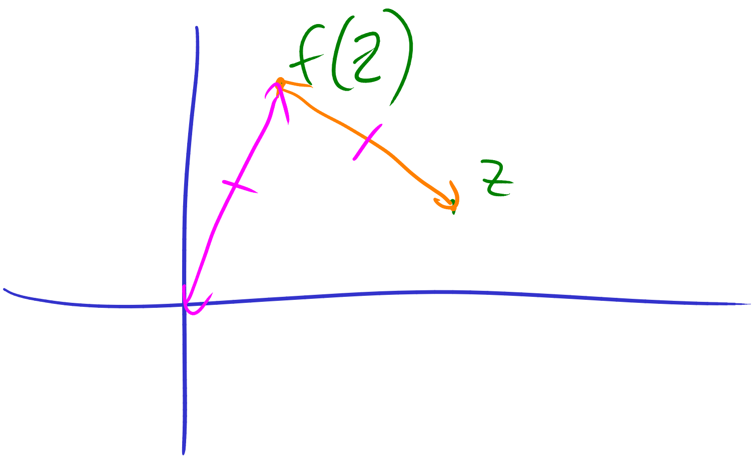
$a^2+b^2=8$

$$f(z) = (a+bi)z$$

$$b^2 = 8 - a^2 = 8 - \left(\frac{1}{2}\right)^2 = \frac{255}{4}$$

$\Rightarrow 255+4=259$

$$|u||v| = |uv|$$



$$|f(z)| = |f(z) - z|$$

$$|(a+bi)z| = |(a+bi)z - z|$$

$$|a+bi|\cancel{|z|} = |a+bi-1|\cancel{|z|}$$

$$|a+bi| = |a+bi-1|$$

$$a^2 + b^2 = (a-1)^2 + b^2$$

$$a^2 = (a-1)^2$$

$$a = \pm(a-1)$$

$$a = -(a-1)$$

$$a = -a+1$$

$$2a = 1 \Rightarrow a = \frac{1}{2}$$

9. There is a complex number z with imaginary part 164 and a positive integer n such that $\frac{z}{z+n} = 4i$, find n.

$$z = a + 164i$$

$$\frac{a+164i}{a+164i+n} = 4i$$

$$a + 164i = 4i(a + 164i + n)$$

$$a + 164i = 4ai - 656 + 4ni$$

$$a = -656$$

$$164 = 4a + 4n$$

$$\frac{164 - 4a}{4} = n$$

$$41 - a = n$$

$$41 + 656 = 697$$

10. Find c if a, b, and c are positive integers which satisfy $c = (a+bi)^3 - 107i$.

$$c = (a+bi)(a+bi)(a+bi) - 107i$$

$$c = (a^2 + 2abi - b^2)(a+bi) - 107i$$

$$c = a^3 + 2a^2bi - ab^2 + a^2bi - 2ab^2 - b^3i - 107i$$

$$c = \underbrace{a^3 - 3ab^2}_{} + \underbrace{3a^2bi - b^3i - 107i}_{}$$

$$c = a^3 - 3ab^2 = 216 - 3\cdot 6\cdot 1^2 = \boxed{198}$$

$$0 = 3a^2b - b^3 - 107$$

$$107 = 3a^2b - b^3$$

$$107 = b(3a^2 - b^2)$$

$$b = 1 \qquad 107$$

$$3a^2 - 1 = 107$$

$$a^2 = \frac{108}{3} = 36$$

$$a = 6$$