The Ninth Grade Math Competition Class
Factorization
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1. Two non-zero real numbers, a and b, satisfy $ab = a - b$. Find all possible values of $\frac{a}{b} + \frac{b}{a} - ab$.

$$\frac{a}{b} + \frac{b}{a} - ab = \frac{a^2 + b^2 - a^2b^2}{ab}$$

$$= \frac{a^2 + b^2 - (a-b)^2}{ab}$$

$$= \frac{\cancel{a^2} + \cancel{b^2} - \cancel{a^2} + 2\cancel{ab} - \cancel{b^2}}{\cancel{ab}} = 2$$

2. Without a calculator, find the sum of the digits of the number $2003^4 - 1997^4$.

$$2003^4 - 1997^4 = (2003^2 - 1997^2)(2003^2 + 1997^2)$$

$$(2000+3)^2 + (2000-3)^2$$

$$2000^2 + \cancel{2 \cdot 2000 \cdot 3} + 3^2 + 2000^2 - \cancel{2 \cdot 2000 \cdot 3} + 3^2$$

$$= (2003-1997)(2003+1997)(8000{,}018)$$

$$6 \cdot 4000 \cdot 8000{,}018$$

$$(2000+3)^4 - (2000-3)^4$$

$$\cancel{2000^4} + \binom{4}{1} 2000^3 \cdot 3 + \cancel{\binom{4}{2} 2000^2 \cdot 3^2} + \binom{4}{3} 2000 \cdot 3^3 + \cancel{3^4}$$

$$\cancel{-2000^4} + \binom{4}{1} 2000^3 \cdot 3 - \cancel{\binom{4}{2} 2000^2 \cdot 3^2} + \binom{4}{3} 2000 \cdot 3^3 - \cancel{3^4}$$

$$8 \cdot 2000^3 \cdot 3 + 4 \cdot 2000 \cdot 3^3$$

3. Express $2^{22}+1$ as the product of two four-digit numbers.

$$4a^4+b^4=$$

$b=1$

$a=2^{10}$

$$\left(2^{22}+1+2\cdot 2^{11}\cdot 1\right)-2\cdot 2^{11}\cdot 1$$

a^2+b^2

$(a+b)^2$ $\quad 2ab$

$$\left(2^{11}+1\right)^2-\underbrace{2\cdot 2^{11}\cdot 1}_{2^{12}}$$

$$=\left(2^{11}+1-2^6\right)\left(2^{11}+1+2^6\right)$$

4. Find the length and the width of a rectangle with integer sides whose area is equal to its perimeter.

5. Two different prime numbers between 4 and 18 are chosen. When their sum is subtracted from their product, which of the following numbers could be obtained?

(A) 21 (B) 60 (C) 119 (D) 180 (E) 231

6. m, n are integers such that $m^2 + 3m^2n^2 = 30n^2 + 517$. Find $3m^2n^2$.

7. How many distinct ordered pairs of positive integers (m, n) are there so that the sum of the reciprocals of m and n is $\frac{1}{4}$?

8. Find all prime factors of $3^{18} - 2^{18}$

9. An $m \times n \times p$ rectangular box has half the volume of an $(m+2) \times (n+2) \times (p+2)$ rectangular box, where $m, n,$ and p are integers, and $m \leq n \leq p$. What is the largest possible value of p?

10. How many non-congruent right triangles with positive integer leg lengths have areas that are numerically equal to 3 times their perimeters?

11. The integer N is positive. There are exactly 2005 pairs (x, y) of positive integers satisfying:

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{N}$$

Prove that N is a perfect square.