The Ninth Grade Math Competition Class
Quadratic Equations and Vieta
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1. Let a and b denote the solutions of $18x^2 + 3x - 28 = 0$, find the value of $(a-1)(b-1)$.

$$a+b = -\frac{3}{18} = -\frac{1}{6}$$

$$-a-b = \frac{1}{6}$$

$$ab = \frac{-28}{18} = \frac{-14}{9}$$

$$(a-1)(b-1) = \underbrace{ab}_{\frac{-14}{9}} \underbrace{-a-b}_{\frac{1}{6}} + 1$$

$$-\frac{14}{9} + \frac{1}{6} + 1 = \boxed{-\frac{7}{18}}$$

$$ab = \frac{2}{1} = 2 \qquad (a+b)^2 = m^2$$

$$a+b = -\frac{-m}{1} = m \qquad a^2+b^2+2ab = m^2$$

$$a^2+b^2 = m^2-4$$

2. Let a and b be the roots of the equation $x^2 - mx + 2 = 0$. Suppose that $a + \frac{1}{b}$ and $b + \frac{1}{a}$ are roots of the equation $x^2 - px + q = 0$, find q.

$$\frac{q}{1} = \left(a+\frac{1}{b}\right)\left(b+\frac{1}{a}\right) = \underbrace{ab}_{2} + \frac{a}{b} + \frac{b}{a} + \frac{1}{\underbrace{ab}_{2}}$$

$$= \frac{5}{2} + \frac{a}{b} + \frac{b}{a}$$

$$= \frac{5}{2} + \frac{\overbrace{a^2+b^2}^{m^2-4}}{\underbrace{ab}_{2}}$$

$$= \frac{5}{2} + \frac{m^2-4}{2}$$

$$= \boxed{\frac{1}{2} + \frac{m^2}{2}}$$

3. Let p, q and r be constants. One sulution to the equation $(x-p)(x-q)=(r-p)(r-q)$ is $x=r$. Find the other solution in terms of p, q and r.

$x=y$

$$x^2 - px - qx + \cancel{pq} = r^2 - pr - qr + \cancel{pq}$$

$$x^2 - (p+q)x - r^2 + pr + qr = 0$$

$$r + y = -\frac{-(p+q)}{1} = p+q$$

$$y = \boxed{p+q-r}$$

4. If m and n are the roots of $x^2 + mx + n = 0$, where $m \neq 0$ and $n \neq 0$, then what number does $m + n$ equal?

$$m+n = -\frac{m}{1} = -\frac{1}{1} = -1$$

$$mn = \frac{n}{1} = n$$

$$mn = n$$

$$m = 1$$

5. For what values of k does the equation $\frac{x-1}{x-2} = \frac{x-k}{x-6}$ have no solution for x?

$x = 2, 6$

$$(x-1)(x-6) = (x-2)(x-k)$$

$$\cancel{x^2} - x - 6x + 6 = \cancel{x^2} - 2x - kx + 2k$$

$$-5x + kx = 2k - 6$$

$$x = \frac{2k-6}{-5+k}$$

$$\Rightarrow k = 5$$

$$\frac{2k-6}{-5+k} = 2 \qquad \cancel{2k} - 6 = -10 + \cancel{2k}$$

$$\frac{2k-6}{-5+k} = 6 \qquad 2k - 6 = -30 + 6k$$

$$24 = 4k \Rightarrow k = 6$$

6. Find all solutions to $2w^4 - 5w^2 + 2 = 0$.

$y = w^2$

$2y^2 - 5y + 2 = 0$

$(2y-1)(y-2) = 0$

$y = \frac{1}{2}, 2$

$\frac{1}{2} = w^2$ $\qquad$ $2 = w^2$

$w = \pm\frac{1}{\sqrt{2}}$ $\qquad$ $w = \pm\sqrt{2}$

7. Find the value of $\sqrt{90+\sqrt{90+\sqrt{90+\cdots}}}$.

$$x = \sqrt{90+\underbrace{\sqrt{90+\sqrt{90+\cdots}}}_{x}}$$

$$x = \sqrt{90+x}$$

$$x^2 = 90 + x$$

$$x^2 - 90 - x = 0$$

$$(x-10)(x+9) = 0$$

$$x = 10, -9$$

8. Let a and b be the roots of $x^2 - 3x - 1 = 0$. Try to solve the following problems without finding a and b, it will be easier that way, anyway.

- Find a quadratic equations whose roots are a^2 and b^2.
- Compute $\frac{1}{a+1} + \frac{1}{b+1}$.

$x^2 + cx + d = 0$

$\frac{c}{1} = -(a^2+b^2)$

$\frac{d}{1} = a^2b^2 = 1$

$-\frac{-3}{1} = a+b = 3$

$\frac{-1}{1} = ab = -1$

$(a+b)^2 = 3^2$

$c = -(a^2+b^2) = -11$

$a^2 + 2ab + b^2 = 9$

$x^2 - 11x + 1 = 0$

$a^2 + b^2 = 11$

$$\frac{1}{a+1} + \frac{1}{b+1} = \frac{a+1+b+1}{\underbrace{(a+1)(b+1)}_{ab+a+b+1}} = \frac{3+2}{-1+3+1} = \frac{5}{3}$$

9. For some integer a, the equation $1988x^2 + ax + 8891 = 0$, and $8891x^2 + ax + 1988 = 0$ share a common root. Find a.

$x = r$

$1988r^2 + ar + 8891 = 0$

$8891r^2 + ar + 1988 = 0$

$6903r^2 - 6903 = 0$

$r^2 - 1 = 0$

$r^2 = 1$ $r = \pm 1$

$r = 1$

$1988 + a + 8891 = 0$

$a = -10879$

$r = -1$

$1988 - a + 8891 = 0$

$a = 10879$

r, s

10. The product of the roots of the quadratic $6x^2 + cx + 4$ is 2 greater than the sum of the roots, and c is a constant. What is c?

$rs = 2 + r + s$

$rs = \frac{4}{6}$

$r + s = -\frac{c}{6}$

$\frac{4}{6} = 2 - \frac{c}{6}$

$\frac{c}{6} = \frac{4}{3}$

$c = 8$

11. Let a,b, and c be the roots of $x^3 - 3x^2 + 1$.

- Find a polynomial whose roots are $a + 3$, $b + 3$ and $c + 3$.
- Find a polynomial whose roots are $\frac{1}{a+3}$, $\frac{1}{b+3}$, and $\frac{1}{c+3}$.
- Compute $\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}$.
- Find a polynomial whose roots are a^2, b^2 and c^2.
- Find a recurrence relation for $x_n = a^n + b^n + c^n$, and use it to compute $a^5 + b^5 + c^5$.