The Ninth Grade Math Competition Class

Radical Expressions and Rationalizing Denominators Problems

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1. Find $\sqrt{9+\sqrt{56}} - \sqrt{9-\sqrt{56}}$.

$\sqrt{a}+\sqrt{b} \quad \sqrt{a}-\sqrt{b}$

$\sqrt{7}+\sqrt{2} - (\sqrt{7}-\sqrt{2}) = 2\sqrt{2}$

$\sqrt{9+\sqrt{56}} = \sqrt{a}+\sqrt{b}$

$9+\sqrt{56} = a+b+2\sqrt{ab}$

$9 = \overset{7}{a}+\overset{2}{b}$

$\sqrt{56} = \sqrt{4}\sqrt{ab}$

$14 = ab$

2. Rationalize the denominator of $\frac{1}{2-\sqrt[3]{2}}$.

$$(a-b)(a^2+ab+b^2)=a^3-b^3$$

$$(a+b)(a^2-ab+b^2)=a^3+b^3$$

$$\frac{1}{2-\sqrt[3]{2}}\,\frac{2^2+2\sqrt[3]{2}+\sqrt[3]{4}}{(2^2+2\sqrt[3]{2}+\sqrt[3]{4})}=\frac{4+2\sqrt[3]{2}+\sqrt[3]{4}}{8-2}=\boxed{\frac{2}{3}+\frac{\sqrt[3]{2}}{3}+\frac{\sqrt[3]{4}}{6}}$$

$$(a-b)(a^2+ab+b^2)=a^3-b^3$$

3. Rationalize the following denominator $\frac{8}{\sqrt{15}-\sqrt{7}}$.

$$\frac{8}{\sqrt{15}-\sqrt{7}} \cdot \frac{\sqrt{15}+\sqrt{7}}{\sqrt{15}+\sqrt{7}} = \frac{\cancel{8}\left(\sqrt{15}+\sqrt{7}\right)}{\underbrace{15-7}_{\cancel{8}}} = \boxed{\sqrt{15}+\sqrt{7}}$$

4. In how many real values of x is $\sqrt{120 - \sqrt{x}}$ an integer?

$\sqrt{120-\sqrt{x}} = k$

$120 - \sqrt{x} = k^2$

$120 - k^2 = \sqrt{x}$

$k=1$

$114 = \sqrt{x}$

$x = 114^2$

$x = (120-k^2)^2$

0
1
4
9
⋮
100
121

$\boxed{11}$

5. Let $a^2 = \frac{4}{11}$, $b^2 = \frac{(2+\sqrt{5})^2}{11}$, where a is a negative real number and b is a positive real number. If $(a+b)^3$ can be expressed in the simplified form $\frac{x\sqrt{y}}{z}$, where x, y, z are positive integers. Find $x + y + z$.

$$a^2 = \frac{4}{11}$$

$$a = \pm\sqrt{\frac{4}{11}}$$

$$b^2 = \frac{(2+\sqrt{5})^2}{11}$$

$$b = \pm\frac{2+\sqrt{5}}{\sqrt{11}}$$

$$b = \frac{2+\sqrt{5}}{\sqrt{11}}$$

$$a = -\sqrt{\frac{4}{11}} = -\frac{\sqrt{4}}{\sqrt{11}} = -\frac{2}{\sqrt{11}}$$

$$(a+b)^3 = \left(-\frac{2}{\sqrt{11}} + \frac{2+\sqrt{5}}{\sqrt{11}}\right)^3 = \left(\frac{\cancel{-2+2}+\sqrt{5}}{\sqrt{11}}\right)^3$$

$$= \left(\frac{\sqrt{5}}{\sqrt{4}}\right)^3 = \frac{\sqrt{125}}{\sqrt{1331}}$$

$$= \frac{5\sqrt{5}}{11\sqrt{11}} \cdot \frac{\sqrt{11}}{\sqrt{4}} = \frac{5\sqrt{55}}{121}$$

$$5 + 55 + 121 = \boxed{181}$$

6. Rationalize the denominator of $\frac{1}{\sqrt[3]{2}+\sqrt[3]{16}}$.

$$\frac{1}{\sqrt[3]{2}+\sqrt[3]{8}\cdot\sqrt[3]{2}} = \frac{1}{\sqrt[3]{2}+2\sqrt[3]{2}} = \frac{1}{3\sqrt[3]{2}}\cdot\frac{\sqrt[3]{4}}{\sqrt[3]{4}} = \boxed{\frac{\sqrt[3]{4}}{6}}$$

$-6 = 2\sqrt{9}$

7. What is the product of the real roots of the equation $\underbrace{x^2 + 18x + 30}_{y-15} = 2\sqrt{\underbrace{x^2 + 18x + 45}_{y}}$.

$\boxed{20}$

$10 = 2\sqrt{25}$

$\cancel{x^2+18x+45=9}$

$\cancel{x^2+18x+36=0}$

$x^2+18x+45=25$

$x^2+18x+20=0$

$y-15 = 2\sqrt{y}$

$y^2-30y+225 = 4y$

$y^2-34y+225=0$

$(y-25)(y-9)=0$

$\cancel{y=9}, 25$

8. Determine the rational number $\frac{a}{b}$ in lowest terms that equal to

$$\frac{1}{\sqrt{2}+2}+\frac{1}{2\sqrt{3}+3\sqrt{2}}+\frac{1}{3\sqrt{4}+4\sqrt{3}}+\cdots+\frac{1}{(2013^2-1)\sqrt{2013^2}+2013^2\sqrt{2013^2-1}}$$

$$\underbrace{\frac{1}{x\sqrt{x+1}+(x+1)\sqrt{x}}}_{a+b}\;\underbrace{\frac{(x\sqrt{x+1}-(x+1)\sqrt{x})}{(x\sqrt{x+1}-(x+1)\sqrt{x})}}_{a-b} = \underbrace{\frac{x\sqrt{x+1}-(x+1)\sqrt{x}}{x^2(x+1)-(x+1)^2x}}_{a^2-b^2}$$

$$= \frac{x\sqrt{x+1}-(x+1)\sqrt{x}}{x^3+x^2-x^3-2x^2+x} = \frac{x\sqrt{x+1}-(x+1)\sqrt{x}}{-x^2+x}$$

$$= \frac{x\sqrt{x+1}-(x+1)\sqrt{x}}{-x(x+1)} = \frac{\cancel{x}\sqrt{x+1}}{-\cancel{x}(x+1)} - \frac{\cancel{(x+1)}\sqrt{x}}{-x\cancel{(x+1)}}$$

$$\boxed{= -\frac{\sqrt{x+1}}{x+1}+\frac{\sqrt{x}}{x}}$$

$$-\cancel{\frac{\sqrt{2}}{2}}+\frac{\sqrt{1}}{1}-\cancel{\frac{\sqrt{3}}{3}}+\cancel{\frac{\sqrt{2}}{2}}-\cancel{\frac{\sqrt{4}}{4}}+\cancel{\frac{\sqrt{3}}{3}}-\cdots\cancel{}$$

$$-\frac{\sqrt{2013^2}}{2013^2}+\cancel{\frac{\sqrt{2013^2-1}}{2013^2-1}}$$

$$\frac{1}{1}-\frac{2013}{2013^2}=1-\frac{1}{2013}=\boxed{\frac{2012}{2013}}$$